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Part 2 of Course I includes a study of set theory, transformations of the plane; properties of lines, planes, line segments and angles; elementary number theory; and rational numbers. Decimal fractions, ratio and proportion, percent, and presenting data using graphs are also presented. (FL)

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UNIFIED MODERN MATHEMATICS

COURSE I

PART II

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Secondary School
Mathematics
Curriculum Improvement Study

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UNIFIED MODERN MATHEMATICS

COURSE I

PART II

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CHAPTER 8

SETS AND RELATIONS

8.1 Sets

Our everyday speech abounds with collective nouns such as herd, company, swarm, class, litter, collection, bunch, etc. Examples which use these collective nouns include the following: a herd of cattle, a company of soldiers, a swarm of bees, a class of students, a litter of kittens, a collection of stamps, a bunch of bananas.

It is also possible to find examples which use collective nouns which may be unfamiliar to you such as the following: a gam of whales, a pod of seals, a glitter of butterflies, a singular of boars, a gaggle of geese, a hutch of rabbits, an army of ants, a murmuration of starlings, a jubilation of skylarks, and a pride of lions.

In each of the above examples we see how a word, such as herd, class, pride, etc., is used to denote a collection of several objects assembled together and thought of as a unit. Each of the above collections is well-defined. By this we mean that we can determine if a given object does or does not belong to the specific collection being considered.

In mathematics we use the collective noun set to indicate a well-defined collection. The objects in sets can be literally anything: numbers, points, lines, people, letters, cities, etc. These objects in sets are called the elements or members of the set. Terms such as "set" and "element" are part of the basic language used in the study of all branches of mathematics. In this chapter, we will concentrate on terms and concepts dealing with sets and relations between sets.

Here are ten particular examples of sets.

Example 1: The numbers 1,2,3,4,5 and 6.

Example 2: The solution set of the open sentence $2 + 5 = x$ in $(W,+)$.

Example 3: The "primary" colors red, yellow and blue.

Example 4: The states in the U.S.A. whose names begin with the letter "M."

Example 5: The numbers 1,2,3,4,6,8,12, and 24.

Example 6: The states in the U.S.A. for which the names of both the state and its capital city begin with the same letter.

Example 7: The numbers -2, -1, 0, 1, and 2.

Example 8: The set of whole numbers which are both even and odd.

Example 9: The numbers 1, 3 and 5.

Example 10: The outcome set for the tossing of a die.

Notice that the sets in the odd numbered examples above are defined by actually listing the elements in the set; and the sets in the even numbered examples are defined by stating properties which can be used to determine if a particular object is or is not an element of the set.

Sets will usually be denoted by capital letters,

$$A, B, X, Y, \ldots$$

Recall that we used "W" to denote the set of whole numbers and

"Z" to denote the set of integers.

There are essentially two ways to specify a particular set. One way, if it is possible, is actually to list the elements in the set. For example,

$$A = \{0, 1, 2, 3\}$$

denotes the set A whose elements are the whole numbers 0, 1, 2, and 3. Note that the names of the elements are separated by commas and enclosed in braces { }. The second way to specify a set is by stating properties which determine or characterize the elements in the set. For example,

$$A = \{x: x \text{ is a whole number and } x < 4\}$$

which is read, "A is the set of all x such that x is a whole number and x is less than 4."

Note: A letter, here "x," is used to denote an arbitrary element of the set; the colon ":" is read "such that."

If an object x is an element of a set A, i.e., A contains x as one of its elements, then we write

$$x \in A.$$

This can also be read "x is a member of A," or "x is in A," or "x belongs to A." To indicate that "x is not an element of set A" we write

$$x \notin A.$$

Thus, for the set A given above we have

$$0 \in A, \quad 1 \in A, \quad 2 \in A, \quad 3 \in A, \text{ and } 4 \notin A.$$

Let us rewrite the Examples 1-10 given earlier, in order to illustrate the above remarks and notation. We shall denote the sets by A_1, A_2, A_3, ..., A_{10} respectively.

Example 1': $A_1 = \{1, 2, 3, 4, 5, 6\}$

Example 2': $A_2 = \{x: x \in W \text{ and } x = 2 + 5\}$

Example 3': A_3 = {red, yellow, blue}

Example 4': A_4 = {x: x is a state in the U.S.A. whose name begins with the letter "M"}

Example 5': $A_5 = \{1, 2, 3, 4, 6, 8, 12, 24\}$

Example 6': A_6 = {x: x is a state in the U.S.A. whose name has the same first letter as the name of its capital city}

Example 7': $A_7 = \{-2, -1, 0, 1, 2\}$

Example 8': A_8 = {x: $x \in W$ and x is even and x is odd}

Example 9': $A_9 = \{1, 3, 5\}$

Example 10': A_{10} = {x: x is an outcome of a toss of a die}

In Example 10' we could also specify the set A_{10} by listing the numbers 1, 2, 3, 4, 5, and 6 as outcomes:

$$A_{10} = \{1, 2, 3, 4, 5, 6\}$$

In Example 8' notice that the set A_8 is in fact the empty set because there are no whole numbers that are both even and odd. The empty set is also called the null set. It is customarily designated by the symbol "$\emptyset$," or by "{ }."

8.2 Exercises

1. Find the eight elements in the set A_4. Refer to a map if necessary.
2. Find the four elements in the set A_6. Refer to a map if necessary.
3. What relationship exists between the sets A_1 and A_{10}?

4. What relationship exists between the sets A_9 and A_1?

5. List the elements in the sets:
   (a) A_2
   (b) A_8

6. Specify the following sets by stating a property which determines or characterizes the elements in the set.
   (a) A_5
   (b) A_7
   (c) A_9

7. List four essentially different sets that you have studied in previous chapters of this book.

8. Find several properties other than the one used in Example 8 which can be used to characterize the null set.

9. Explain why each of the following is true, or is not true.
   (a) $7 \in A_{10}$
   (b) Delaware is an element of set A_8.
   (c) $0 \in A_8$
   (d) $x \notin A_3$

10. State a property that is true of all the sets $A_1 - A_{10}$.

8.3 Set Equality and Subsets

Let $A = \{0, 1, 2, 3\}$
and $B = \{1, 0, 3, 2\}$

Observe that set A and set B contain precisely the same elements although they are not listed in the same order. A and B are really the same set. We shall indicate this fact by writing

$$A = B.$$

Although this is read, "set A is equal to set B," it means that set A and set B contain precisely the same elements and that we do not have two sets but only one.

In general, if "A" denotes a set, and "B" denotes a set, the statement

$$A = B$$

means that "A" and "B" denote the same set. If sets A and B are not the same set then we write

$$A \neq B.$$

Example 1: If $X = \{0,1\}$ and $Y = \{x\colon x \in W \text{ and } x < 2\}$ then we have $X = Y$.

Example 2: If $V = \{\text{red, green, blue}\}$ and $Y = \{\text{green, blue, red}\}$ then $V = Y$. (Note that the order in which the elements are listed is immaterial.)

Example 3: If $V = \{\text{red, green, blue}\}$ and $X = \{x\colon x \text{ is a color in the rainbow}\}$ then $V \neq X$, because there are other colors such as yellow in the rainbow. Yellow is an element of X, but is not an element of V.

Since each of the colors red, green and blue is also a color in the rainbow, it is clear that every element of set V is an element of set X, or that V is a subset of set X, or that set V is contained in set X. We denote the relation " is a subset of " by the symbol " $\subset$." Thus in Example 3, $V \subset X$.

Definition: Set A is a subset of set B, denoted by $A \subset B$, if and only if every element of set A is an element of set B.

Notice that the above definition implies that if $A \subset B$ and $x \in A$, then $x \in B$.

Example 1: Let $A = \{1\}$, $B = \{0, 1, 2\}$, $C = \{3, 4, 5, 6\}$, and $D = \{0, 1, 2, 3, 4, 5\}$. Then we see that $A \subset B$, $B \subset D$, $A \subset D$.

Example 2: Let $X = \{a, b, c\}$ and $Y = \{c, a, b\}$. We see that $X \subset Y$ because every element of X is an element of Y. Furthermore $Y \subset X$.

Notice in Example 1 that C is not a subset of D because C contains the element 6, whereas D does not contain this element. This illustrates

Remark 1: If set A is not a subset of set B, then set A contains at least one element that is not contained in set B.

Notice also that Example 2 shows that $A \subset B$ does not exclude the possibility that $A = B$. In fact, we can make the following general remark concerning how equality of sets is related to the idea of subset:

Remark 2: If A is a set and B is a set, then $A = B$, if and only if $A \subset B$ and $B \subset A$.

Let us illustrate the above statement.

If $A = \{0, 1, 2, 3\}$ and $B = \{1, 0, 3, 2\}$ then clearly $A \subset B$ because every element in set A is also an element in set B. Also, $B \subset A$ because every element in set B is also an element in set A. Thus, we conclude that $A = B$.

From the above we see that every set has at least one subset, namely, itself. In fact, Remark 3: If A is any set, then $A \subset A$.

We can examine a given set to see what subsets it contains. For example, what subsets may be formed from the set $A = \{2,3\}$? First of all, according to the above remark A is a subset itself. Thus $\{2,3\} \subset A$ or equivalently $\{2,3\} \subset \{2,3\}$. Also it is clear that set A yields two subsets each of which contains a single element. That is

$$\{2\} \subset A \text{ and } \{3\} \subset A.$$

It is curious, but true, that the empty set is a subset of any set. This conclusion is logically forced upon us by Remark 1 above, because if we assume that $\emptyset$ is not a subset of A, then Remark 1 implies that $\emptyset$ contains at least one element that is not an element of A. But $\emptyset$ contains no such element since by definition $\emptyset$ contains no elements. Thus we cannot say that $\emptyset$ is not a subset of A , i.e., $\emptyset$ is a subset of A. Since the above argument would apply to any set A, we conclude with

Remark 4: If A is any set, then $\emptyset \subset A$. Observe that the set $A = \{2,3\}$ has exactly four subsets:

$$\{2,3\}, \{2\}, \{3\} \text{ and } \emptyset.$$

Of these four subsets of A we shall say that $\{2\}$, $\{3\}$ and $\emptyset$ are proper subsets of A and that $\{2,3\}$ is not a proper subset of A. Note that proper subsets of a set do not contain all the elements of the given set. In general we have the following

Definition: A is a proper subset of B, if and only if

$$A \subset B \text{ and } A \neq B.$$

Example 1: Let $K = \{-1, 0, 1\}$. Then each of $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, is a proper subset of K. Note $\{-1, 0, 1\}$ is a subset of K but not a proper subset of K.

Example 2: Let X be any set except the empty set. Then because we know the $\emptyset \subset X$ by Remark 4, and because we are given $\emptyset \neq X$, we conclude that $\emptyset$ is a proper subset of X.

8.4 Exercises

1. Let $G = \{0, 1, 3, 7\}$ and $H = \{7, 1, 0, 3\}$. Explain why $G = H$, or why not.
2. If $G = \{0, 1, 3, 7\}$ and $L = \{x: x \in W, x < 10\}$ then explain why:

   (a) $G \subset L$ (b) $G \neq L$
3. Mr. Jones has five children: Tom, Joan, Judy, Harry and Dick. Let B = {Tom, Dick, Harry}, G = {Judy, Joan} R = {Tom, Joan, Harry, Judy}.
   - (a) Explain why B is a subset of R, or why not.
   - (b) Explain why G is a subset of R, or why not.
4. Let $E = \{x: x \in W \text{ and } x \text{ is even}\}$ and $P = \{x: x \text{ is a positive power of } 2\}$, i.e., $P = \{2, 4, 8, 16, \ldots\}$.

   Explain why the following are or are not true:

   (a) $P \subset E$ (d) $100 \in E$

   (b) $P = E$ (e) $100 \in P$

   (c) $0 \in E$ (f) $E \subset P$

   (g) $\emptyset \subset P$

5. Let $A = \{5\}$.
   (a) List all of A's subsets.
   (b) List all of A's proper subsets.
6. Let $B = \{5,7,9\}$.
   (a) List all of B's subsets.
   (b) List all of B's proper subsets.
7. Using the data obtained in Exercises 5 and 6 above make a conjecture concerning
   (a) the number of subsets in a set containing 4 elements.
   (b) the number of proper subsets in a set containing 4 elements.
   (c) the number of subsets in a set containing 5 elements.
   (d) the number of proper subsets in a set containing 5 elements.
   (e) the number of subsets in a set containing $\underline{n}$ elements.
8. What can we conclude if we know that A is a subset of B but that B is not a subset of A?
9. What conclusions, if any, can you draw from the following?
   (a) $X \subset Y$ and $Y \subset Z$.
   (b) $R \subset S$ and $T \subset R$.
   (c) $M \subset N$ and $N \subset Q$.
   (d) $X \subset G$, $Y \subset T$, and $T \subset X$.
   (e) $A \subset Q$, $Q \subset R$, and $R \subset A$.
   (f) $P \subset Q$, and $R \subset Q$.
10. Let $A = \{p,q,r\}$. Explain why the following are correct or incorrect in the use of "$\subset$" and "$\in$."
    (a) $p \in A$ (d) $A \subset A$

(b) $p \subset A$ (e) $\{p\} \in A$

(c) $\{p\} \subset A$ (f) $\emptyset \subset A$

11. Which of the following sets are the same?

(a) $\{x$: x is a letter in the word "follow"$\}$

(b) $\{x$: x is a letter in the word "wolf."$\}$

(c) the set of letters in the word "flow."

12. Explain why the sets $\emptyset$ and $\{\emptyset\}$ are different sets.

13. Let $X \subset Y$ and $Y \subset Z$. Assume $x \in X$, $y \in Y$, $z \in Z$, and also assume $p \notin X$, $q \notin Y$, $r \notin Z$.

Which of the following must be true? Explain.

(a) $x \in Z$ (c) $z \notin X$ (e) $q \notin X$

(b) $y \in X$ (d) $p \in Y$ (f) $r \notin X$

8.5 Universal Set, Subsets and Venn Diagrams

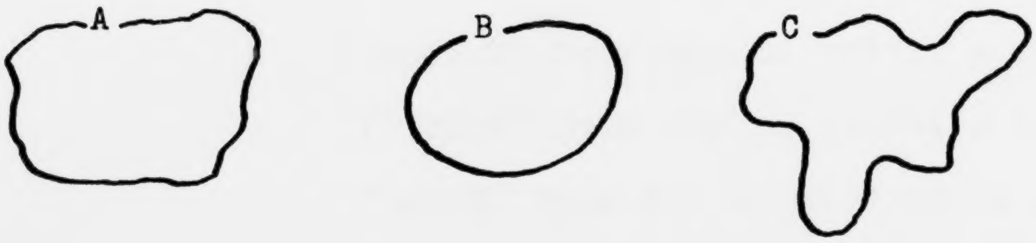
In order to avoid certain logical difficulties, we will assume that in a given discussion the sets being considered are subsets of a set S, called the universal set. We have already seen situations where the idea of a universal set played an important role. For example, in finding solution sets for open sentences we have seen that results depend on the domain or universal set considered.

The solution set of the open sentence

$$3 + x = 2$$

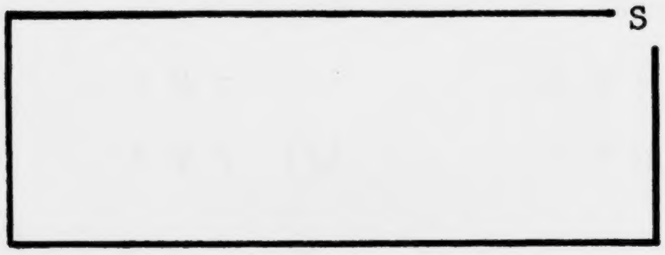
is $\{-1\}$ if the universal set considered is the set Z, whereas it is $\emptyset$ if the universal set is set W.

In order to help visualize our work with sets we shall draw diagrams, called Venn diagrams, which illustrate them. Here we represent a set by a region bounded by a simple closed curve, for example:



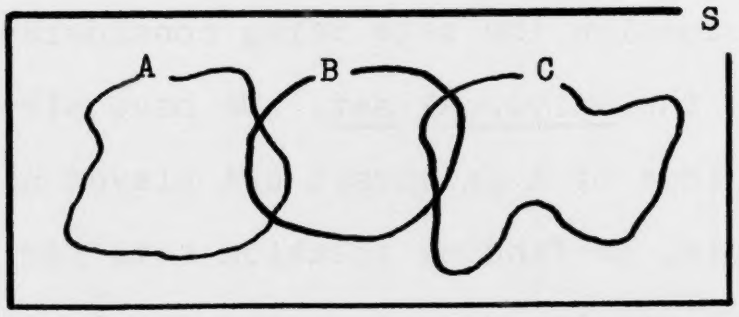
Sets A,B,C

We shall usually indicate the universal set S as a plane region bounded by a rectangle.



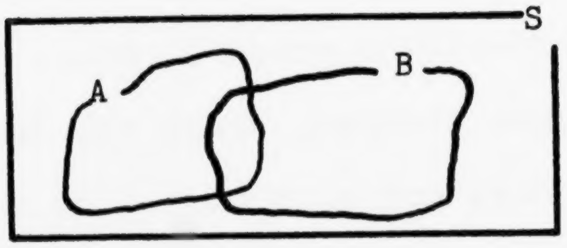
A universal set

Subsets of the universal set will be pictured by regions enclosed within this rectangle.



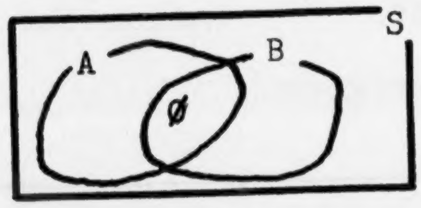
Subsets of a
Universal Set

When we picture sets in this manner we must take care not to confuse the geometric regions with the sets that these regions represent. For example suppose that the universal set S consists of all the students in your school. Suppose the subset A consists of those students who are studying art, and the subset B consists of those students who are studying biology. We can picture these sets by means of a Venn diagram like this:



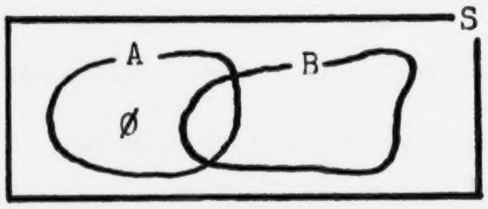
The regions which represent the subsets A and B are drawn so that they appear to overlap. This is done so as to provide a region, namely the overlap, which will represent the subset of those students who take both art and biology. The other portion of the A-region then represents the set of those students in the school who study art but not biology. Similarly the other part of the B-region represents the set of those students in the school who are studying biology but not art. The region of S outside of both A and B represents the set of students not taking art or biology.

Now it may happen that one or more of these regions actually represents an empty set. For example, suppose there are no students in the school who take both art and biology. In that case the overlap region represents the null set. This information can be shown on the Venn diagram by placing the symbol "Ø" inside the overlap region:



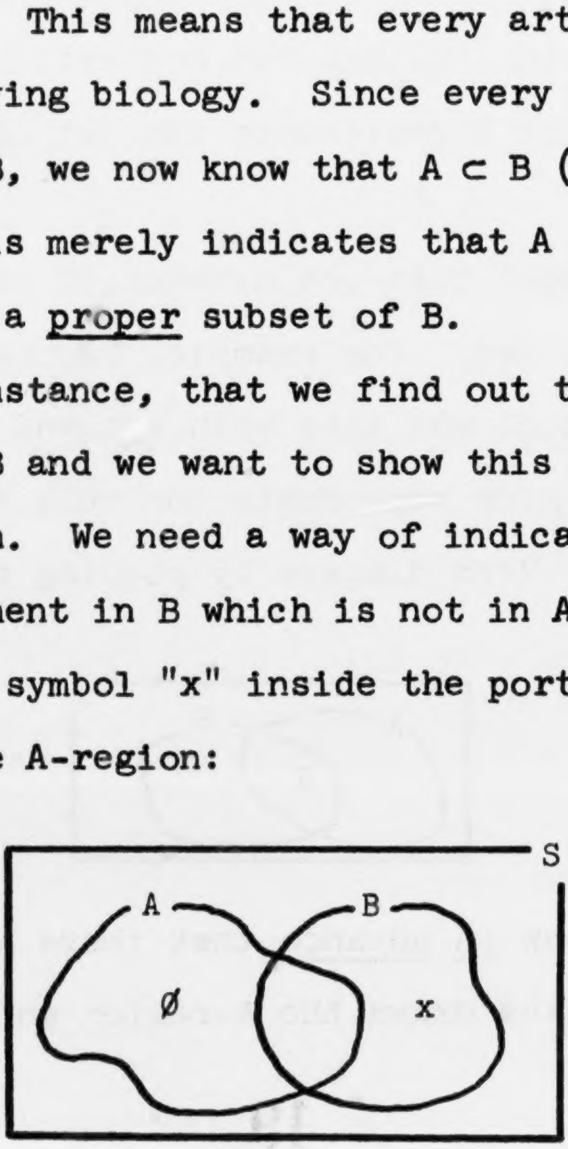
Of course, if we knew in advance that there were no members in this set we could have drawn the A-region and B-region so that they did not overlap, but very often when dealing with sets we may not have such information ahead of time. It is therefore better to draw the Venn diagram, where the regions appear to overlap, with the understanding that one or more of these regions may, on further investigation, actually turn out to be empty.

As another example, suppose that our Venn diagram for the students taking art or biology looked like this:



The symbol "Ø" now indicates that there are no elements of A "outside" of set B. This means that every art student in the school is also studying biology. Since every element of A is also an element of B, we now know that A ⊂ B (A is a subset of B). Notice that this merely indicates that A is a subset of B, but not necessarily a proper subset of B.

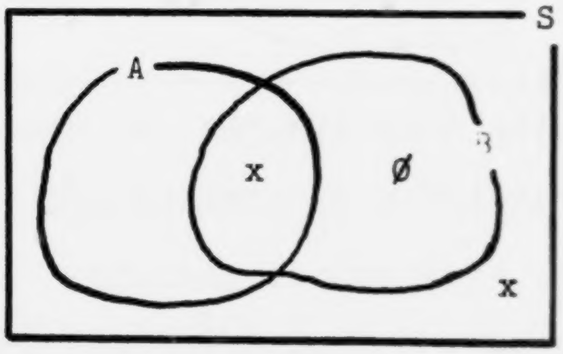
Suppose, for instance, that we find out that A is actually a proper subset of B and we want to show this information on the earlier Venn diagram. We need a way of indicating that there is at least one element in B which is not in A. We shall do this by placing the symbol "x" inside the portion of the B-region which is outside the A-region:



This shows that A is not only a subset of B, but also that there is at least one element in B which is not in A.

Skill in drawing and interpreting Venn diagrams can be helpful so let us study several further illustrations.

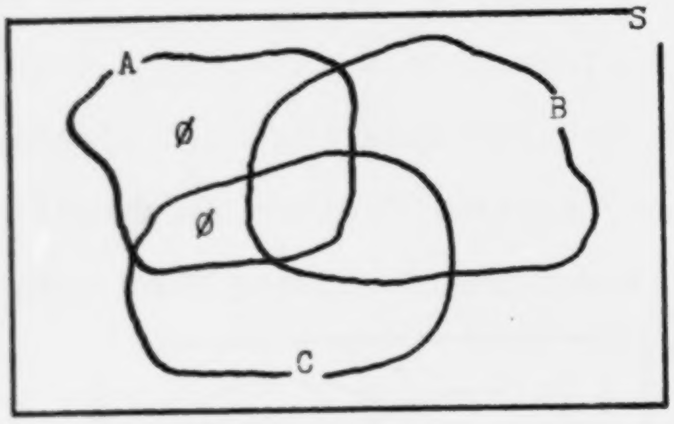
Example 1: Interpret the following Venn diagram:



First of all, $B \subset A$, because there are no elements of B "outside" A. Moreover, set B is not empty, as is indicated by the "x" in the other portion of the B-region. This "x" also shows that A is not empty. We cannot tell if B is a proper subset of A, because there is no information indicated for the portion of the A-region which is outside the B-region. However, the other symbol "x" outside both regions, indicates that there is at least one element in S which is not in A. Hence A is a proper subset of S. We can summarize all this information briefly, as follows:

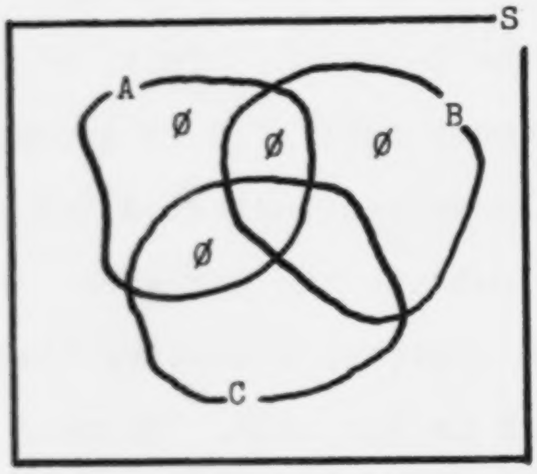
$$\emptyset \subset B \subset A \subset S, \quad A \neq \emptyset, \quad A \neq S, \quad \text{and } B \neq \emptyset.$$

Example 2: Draw a Venn diagram which will show that A is a non-empty subset of B, and B is a proper subset of C. (in symbols: $A \neq \emptyset$, $A \subset B$, $B \subset C$, $B \neq C$) We can do this in more than one way. We can start with a general diagram for the three sets. On it we indicate first that $A \subset B$.

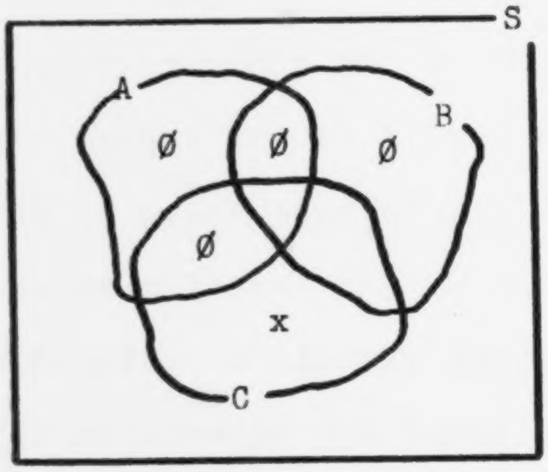

S
A
B
C
$\emptyset$
$\emptyset$

$A \subset B$

Then, on the same diagram, we mark the information that $B \subset C$ by placing additional symbols "$\emptyset$" in the appropriate regions. To show B is a proper subset of C we place an "x" in the region of C which is outside of B, indicating $B \neq C$.

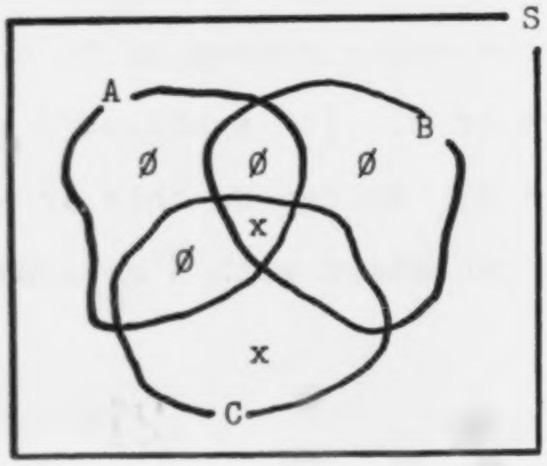

S
A
B
C
$\emptyset$
$\emptyset$
$\emptyset$
$\emptyset$

$A \subset B$ and $B \subset C$


S
A
B
C
$\emptyset$
$\emptyset$
$\emptyset$
$\emptyset$
x

$A \subset B$ and $B \subset C$ and $B \neq C$

Finally, to show that $A \neq \emptyset$ we enter the symbol "x" in the remaining region of set A.

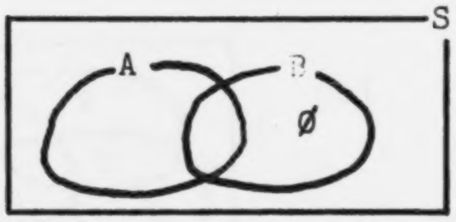

S
A
B
C
$\emptyset$
$\emptyset$
$\emptyset$
$\emptyset$
x
x

Notice that the conditions of the problem forced us to place an "x" in the "center" region. This shows that there is an element that is not only in A, but also in B and in C. We know now that there are at least two elements in C, one "within" B and one "outside of" B.
On the other hand, we still do not know whether or not the remaining two regions of the Venn diagram represent empty or non-empty sets. (These are (1) the region "inside" B, "inside" C and "outside" A, and (2) the region "outside" A, B and C.) We simply have no information about them.

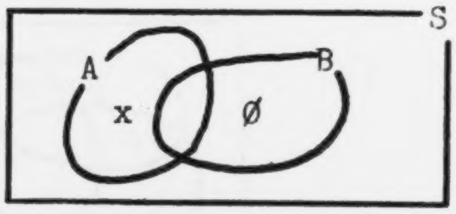
8.6 Exercises

1. What information about sets A and B is revealed by each of the following Venn diagrams?

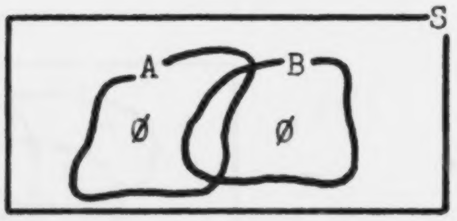
(a)



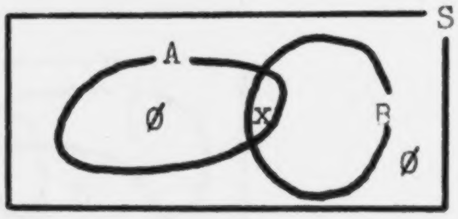
(b)



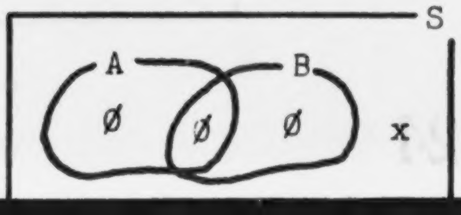
(c)



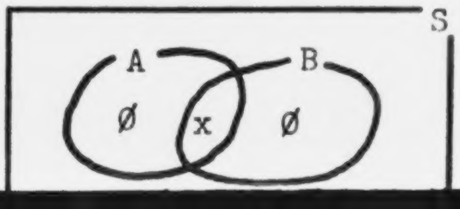
(d)



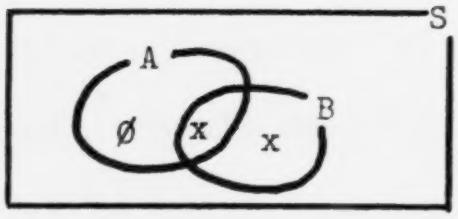
(e)



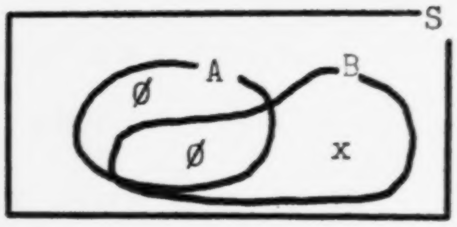
(f)



(g)



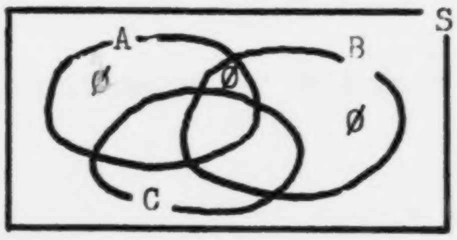
(h)



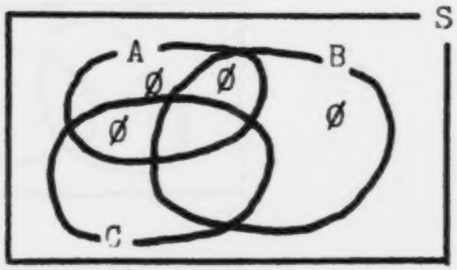
2. Draw a Venn diagram to show each of the following for subsets A and B of a universal set S:
   (a) $B \subset A$ and $B \neq \emptyset$
   (b) $A \subset B$ and $A \neq B$
   (c) $A \subset B$ and $B = S$
   (d) $A \subset B$ and $B \neq S$
   (e) $0 \subset A \subset B \subset S$
   (f) $B \subset A$ and $B \neq \emptyset$ and $A = S$

3. Interpret each of the following Venn diagrams for subsets A, B, C of a universal set S.

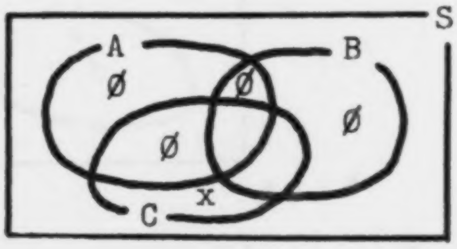
(a)



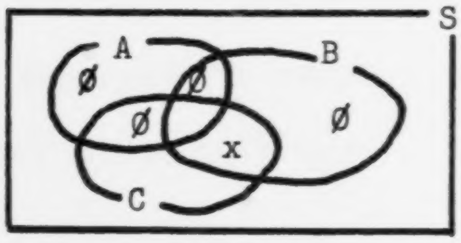
(b)



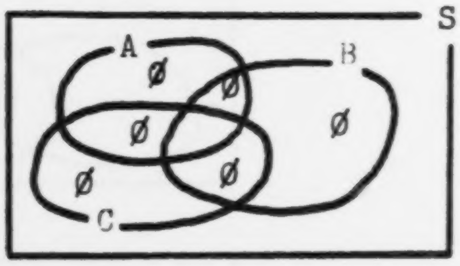
(c)



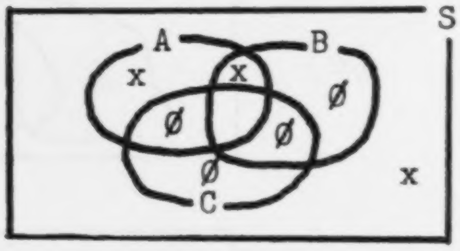
(d)



(e)


S
A
B
C
Ø
Ø
Ø
Ø
Ø
Ø

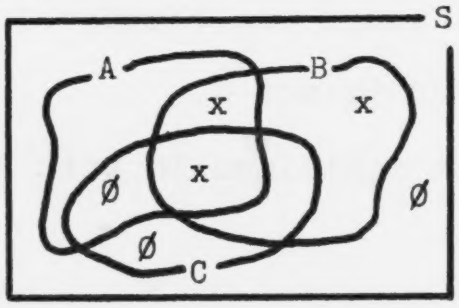
(f)


S
A
B
C
x
x
Ø
Ø
Ø
Ø
x

4. Draw a Venn diagram to illustrate each of the following situations for subsets A, B, C of a universal set S:
   - (a) $A \subset B \subset C$
   - (b) $A = B$ and $C \neq \emptyset$
   - (c) $A = B$ and $C = \emptyset$
   - (d) $\emptyset \subset A \subset B = C$
   - (e) $\emptyset \subset B = A \subset C \subset S$

5. Draw a Venn diagram which shows the following for subsets A, B of a universal set S:
   - (a) There is at least one element in A which is not in B and at least one element in B which is not in A.
   - (b) There is at least one element which is a member of both A and B.
   - (c) There are no elements in both A and B but there is at least one element in S which is neither in A nor in B.
   - (d) Every element of S is either in A or in B.

6. The following Venn diagram applies to subsets A, B, C of a universal set S:



For each of the following statements, write

"YES" if the statement must be true;

"NO" if the statement must be false;

"MAYBE" if the statement may be true or may be false.

- (a) There is at least one element which is contained in all three sets A, B, C.

(b) C is a subset of B.

(c) A is a subset of C.

(d) C is a proper subset of B.

(e) There is at least one element in A which is not in B.

(f) There is at least one element in C which is not in B.

(g) There is at least one element in A which is not in C.

(h) A is a subset of B.

(i) There are at least two elements in S which are contained in both A and B.

(j) There is at least one element in S which is not contained in any of the sets A, B, or C.

(k) There is at least one element in set B which is not contained in either A or C.

(l) There is at least one element in set C which is not contained in either A or B.

(m) There is at least one element in set A which is not contained in either B or C.

(n) There is at least one element in both A and B which is not contained in set C.

(o) Each element contained in both A and C is also contained in B.

8.7 Unions, Intersections and Complements

In earlier chapters we considered operations which assigned new numbers to given ordered pairs of numbers. Now we shall consider how new sets can be formed from given sets. There are two important binary operations that we shall define on sets, and one important unary operation. These operations have many uses in subsequent work. In what follows we assume that the sets A and B are subsets of a universal set S.

If A and B are sets we shall define a new set called the union of A and B, denoted by "A ∪ B," as follows:

Definition: A ∪ B is the set that contains those and only those elements each of which belongs either to A or to B (or to both); i.e.,

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$

(Notice that here "or" is used in the sense "and/or.")

Example 1: If $A = \{0,1,2,3\}$ and $B = \{3,4,5\}$, then $A \cup B = \{0,1,2,3,4,5\}$.

Example 2: If V = {red, green, blue, violet, yellow} and X = {violet, indigo, blue, orange} then V ∪ X = {violet, indigo, blue, green, yellow, orange, red}.

Example 3: If W is the set of whole numbers and $A = \{0, 1, 2,3\}$, then $W \cup A = W$.

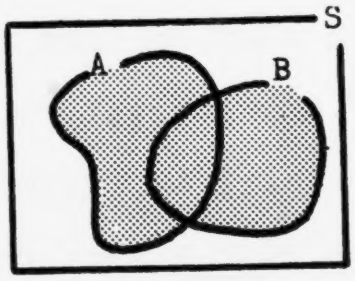
Remark 1: From the definition of A ∪ B we see that

$$A \cup B = B \cup A.$$

Remark 2: Since A ∪ B contains all the elements of A and also contains all the elements of B we can conclude that

$$A \subset (A \cup B) \text{ and } B \subset (A \cup B).$$

In the Venn diagram below we have shaded the region which represents A ∪ B.



A ∪ B is shaded

If A and B are sets we now define a new set called the intersection of A and B, denoted by "A ∩ B," as follows:

Definition: A ∩ B is the set that contains those and only those elements each of which belongs to both A and B, i.e., $A \cap B = \{x: \ x \in A \text{ and } x \in B\}$.

Example 1. If A = {0,1,2,3} and B = {3,4,5}, then A ∩ B = {3}.

Example 2. If V = {red, green, blue, violet, yellow} and X = {violet, indigo, blue, orange} then V ∩ X = {blue, violet}.

Example 3. If W is the set of whole numbers and A = {0,1,2,3}, then W ∩ A = {0,1,2,3} = A.

Example 4. If A = {0,1,2,3}, B = {3,4,5}, and C = {0,3,5}, then (A ∩ B) ∩ C = {3} ∩ {0,3,5} = {3}.

Example 5. If A = {0,1,2,3} and B = {4,5} then A ∩ B = { } = ∅.

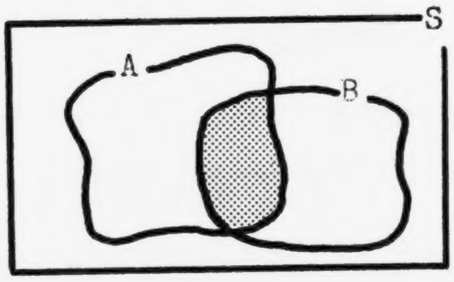
Remark 1: From the definition of A ∩ B we see that

$$A \cap B = B \cap A.$$

Remark 2: If A ∩ B = ∅, as in Example 5, this indicates that sets A and B have no elements in common. In that case we say that A and B are disjoint sets.

Definition: A and B are called disjoint sets if and only if $A \cap B = \emptyset$.

In the Venn diagram below we have shaded the region that represents $A \cap B$.



$A \cap B$ is shaded

Besides obtaining new sets by assigning a new set to a pair of sets it is also useful to define a particular unary operation on every subset of S. If A is a given subset of a given universal set S, we define a new set called the complement of A, denoted by "$\overline{A}$," as follows:

Definition: $\overline{A}$ is the set of all elements of a given universe S that are not contained in A, where A is a subset of S, i.e.,

$$\overline{A} = \{x: x \in S \text{ and } x \notin A\}.$$

Example 1: If $S = \{0,1,2,3,4,5\}$ and $A = \{0,2\}$, then $\overline{A} = \{1,3,4,5\}$.

Example 2: Let $S = W$, that is, the universal set is the set of whole numbers. Let $E = \{x: x \in W$ and x is even$\}$ and $O = \{x: x \in W$ and x is odd$\}$. Then $\overline{E} = O$. That is, the complement of the set of even whole numbers in the set of whole numbers is the set of odd whole numbers. Similarly, $\overline{O} = E$.

Example 3: If $S = \{0,1,2,3,4\}$, $A = \{0,2,4\}$ and $B = \{3,4\}$, then

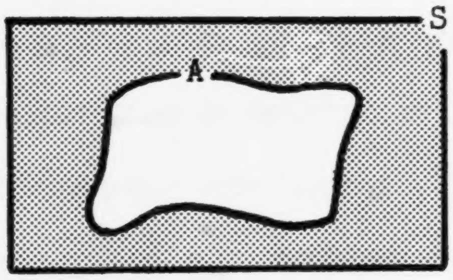
i) $\overline{A} = \{1,3\}$

ii) $\overline{B} = \{0,1,2\}$

iii) Since $A \cap B = \{4\}$ we see that

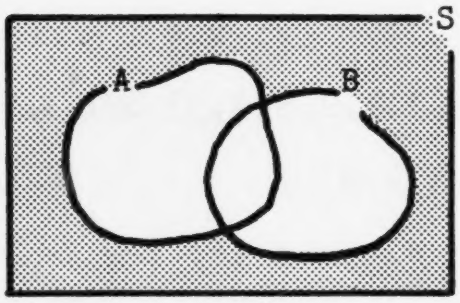
$$\overline{A \cap B} = \overline{\{4\}} = \{0,1,2,3\}$$

The Venn diagram for $\overline{A}$ is given below, i.e., all of S is shaded except A.



$\overline{A}$ is shaded

The Venn diagram for $\overline{A \cup B}$ is given below. Since $\overline{A \cup B}$ is the set consisting of all elements in S that are not in the set $A \cup B$ we shade all of the S-region except the part that represents $A \cup B$.



$\overline{A \cup B}$ is shaded

8.8 Exercises

1. Let the universal set S be the set of all students enrolled in your school. S = {all students}. Let A, B and C be the following subsets of S. A = {all 7th graders}, B = {all boys}, C = {all students who bus to school}.

   Describe in words each of the following:

(a) $\overline{A}$

(b) $\overline{B}$

(c) $A \cap B$

(d) $B \cap C$

(e) $A \cup B$

(f) $B \cup C$

(g) $A \cap \overline{B}$

(h) $\overline{A} \cup B$

(i) $\overline{B} \cap \overline{C}$

(j) $\overline{B} \cup \overline{C}$

(k) $A \cup (B \cup C)$

(l) $A \cap (B \cap C)$

(m) $A \cap (\overline{B} \cup \overline{C})$

(n) $\overline{A} \cup (B \cup \overline{C})$

(o) $\overline{A} \cap (\overline{B \cap C})$

(p) $A \cap (\overline{B \cup C})$

2. Let the universal set S be $S = \{0,1,2,3,4,5,6\ 7,8,9\}$. Further, let $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{2, 3, 5, 7\}$.

Determine the following:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \cup C$

(d) $A \cap C$

(e) $B \cup C$

(f) $B \cap C$

(g) $\overline{A}$

(h) $\overline{B}$

(i) $\overline{C}$

(j) $\overline{A} \cup \overline{C}$

(k) $\overline{A \cap C}$

(l) $\overline{S}$

3. Using the sets in Exercise 2 determine the following:

(a) $(A \cup B) \cup C$

(b) $A \cup (B \cup C)$

(c) $A \cap (B \cap C)$

(d) $(A \cap B) \cap C$

4. Using the sets in Exercise 2 determine the following:

(a) $A \cup (B \cap C)$

(b) $(A \cup B) \cap (A \cup C)$

(c) $A \cap (B \cup C)$

(d) $(A \cap B) \cup (A \cap C)$

5. If $S = \{-4,-3,0,3,4,7,8,16\}$

   $A = \{-4,0,8,16\}$

   $B = \{-3,3,4,8\}$

   $C = \{0,7\}$

   Determine each of the following subsets of S:

   (a) $A \cup B$

   (b) $\overline{A \cup B}$

   (c) $\overline{A} \cap \overline{B}$

   (d) $A \cap B$

   (e) $\overline{A \cap B}$

   (f) $A \cup (A \cap B)$

   (g) $A \cap (A \cup B)$

   (h) $\overline{A} \cup \overline{B}$

   (i) $A \cap (B \cup C)$

   (j) $(A \cup B) \cup (A \cap C)$

   (k) $A \cup (B \cap C)$

   (l) $(A \cup B) \cap (A \cup C)$

   (m) $\overline{A \cup (B \cup C)}$

   (n) $\overline{A} \cap (\overline{B} \cap \overline{C})$

6. Using the data obtained in the above exercises state some conjectures concerning the properties of the operations of union and intersection on any sets A, B, and C. Can you offer any further evidence to support your conjectures?

7. Let N be the set of natural numbers, i.e., the set of whole numbers with zero deleted. Let the universal set be W, that is, the set of whole numbers. Determine if the following are true or false. Explain your answers.

   (a) $N \cup W = W$

   (b) $N \cap W = \{0\}$

   (c) $\overline{N} = \{0\}$

   (d) $\overline{W} = 0$

   (e) $\overline{W} \cup \overline{N} = \emptyset$

   (f) $\overline{W} \cap \overline{N} = \emptyset$

   (g) $\overline{W \cup N} = \emptyset$

   (h) $\overline{W \cap N} = \emptyset$

8. Using the definitions of "subset," "intersection," and "union" write out an argument why the following are true:

   (a) $(A \cap B) \subset A$

   (b) $(A \cap B) \subset A \cup B$.

9. Using your definitions explain why the following are true:

If A is any subset of a universal set S, then:

(a) $A \cup A = A$ (e) $\bar{S} = \emptyset$

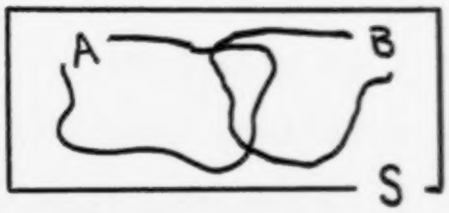
(b) $A \cap A = A$ (f) $\bar{\emptyset} = S$

(c) $A \cup \bar{A} = S$ (g) $A \cup S = S$

(d) $A \cap \bar{A} = \emptyset$ (h) $A \cap \emptyset = \emptyset$

10. If we denote the complement of the complement of set A by "$\bar{\bar{A}}$" determine what set $\bar{\bar{A}}$ is equal to.

11. Copy the Venn diagram below and shade in the set represented by $A \cap \bar{B}$.



12. Let us define a new operation, called the difference of A and B, denoted by "A\B", as follows: If A and B are subsets of a universal set S, then $A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}$.

(a) Determine if $A \backslash B = A \cap \bar{B}$.

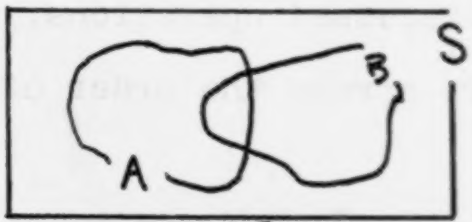
(b) Determine if $A \backslash B = B \backslash A$.

(c) Determine if $(A \backslash B) \subset A$.

(d) Determine the set represented by the union of $A \backslash B$, $A \cap B$, and $B \backslash A$.

(e) Determine the set represented by the intersection of $A \backslash B$ and $B \backslash A$.

13. Copy the Venndiagram below and shade in the set represented by $(A \cap \bar{B}) \cup (\bar{A} \cap B)$.



**14. Let us define a new operation called the *symmetric difference* of A and B, denoted by " $A \Delta B$," as follows:

$$A \Delta B = (A \backslash B) \cup (B \backslash A).$$

(a) Determine if $A \Delta B = (A \cup B) \backslash (A \cap B)$.

(b) Determine if $A \Delta B = \{x: x \in A \text{ or } x \in B, \text{ but } x \notin A \cap B\}$.

(c) Determine if $A \Delta B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$.

(d) Determine what set is represented by $(A \cup B) \backslash (A \Delta B)$.

15. Let the universal set be Z (the set of all integers).

Let A, B, C, D be the following subsets of Z:

$A = \{x: x \text{ is positive}\}$

$B = \{x: x \text{ is negative}\}$

$C = \{x: x \text{ is less than } 10\}$

$D = \{x: x \text{ is greater than } -5\}$

Determine each of the following sets:

(a) $A \cap B$

(b) $A \cup B$

(c) $B \cap C$

(d) $B \cup C$

(e) $C \cup D$

(f) $C \cap D$

(g) $(A \cap C) \cap D$

(h) $A \cap (C \cap D)$

(i) $A \cap (C \cup D)$

(j) $(A \cap C) \cup D$

8.9 Cartesian Product Sets: Relations

In earlier parts of this book you often dealt with the idea of an ordered pair of elements. In many cases you had to distinguish between the pair (a,b) and the pair (b,a). For example, this occured when you discussed operations, mappings, outcome sets, lattices, etc. To stress the order of the elements, one of the elements in the pair was designated as the first element or first coordinate of the pair, and the remaining element was designated as the second element or second coordinate of the pair. We shall now make use of the idea of ordered pair in order to show how a new set can be formed from two given sets.

Let A = {1,2,3} and B = {4,6}. From the sets A and B we form all pairs such that each pair contains an element of A as first element and an element of B as second element. These pairs are (1,4), (1,6), (2,4), (2,6), (3,4), and (3,6). We designate the set of these ordered pairs by "A x B." This new set of ordered pairs is called the Cartesian product of A and B, or simply "A cross B." Thus:

A x B = {(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)}

Note: The set A x B is named after the mathematician Rene Descartes who, in the seventeenth century, studied such sets.)

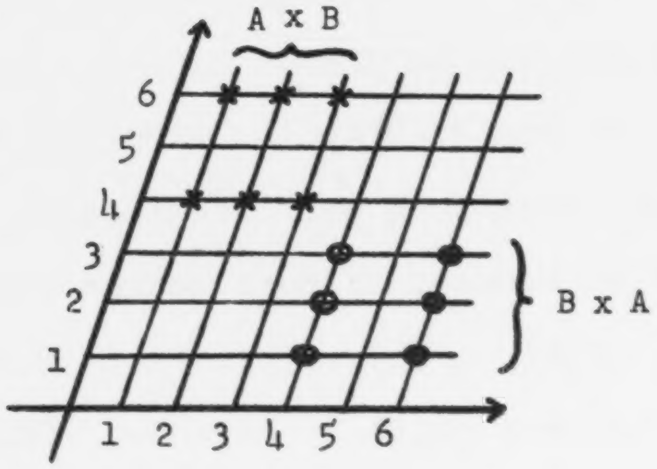
Observe that set A contains three elements, set B contains two elements, and the set A x B contains six elements.

Given the same sets A and B as above we can also form the set B x A We have

B x A = {(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)}

We see that if we reverse the coordinates of each ordered pair in A x B we obtain B x A. It is important to note that although B x A also contains six elements, it is not the same as A x B. In fact, unless A = B, A x B ≠ B x A.

We can illustrate this by graphing the lattice points associated with each of the Cartesian products. In the graph below we see that elements of A x B are represented by points with crosses, whereas the elements of B x A are represented by points with circles.

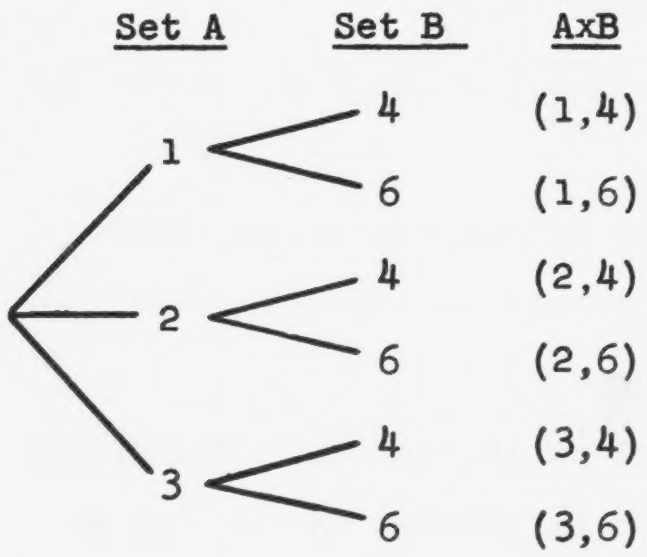


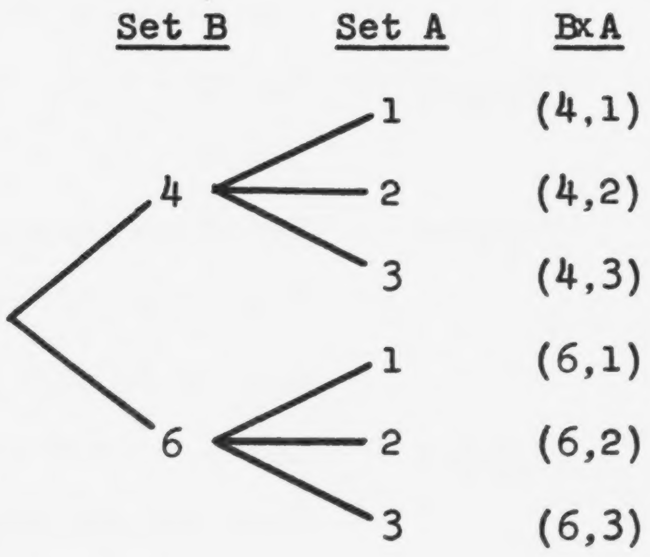
We often form the Cartesian product of a set with itself. Thus, for the given sets A and B we obtain:

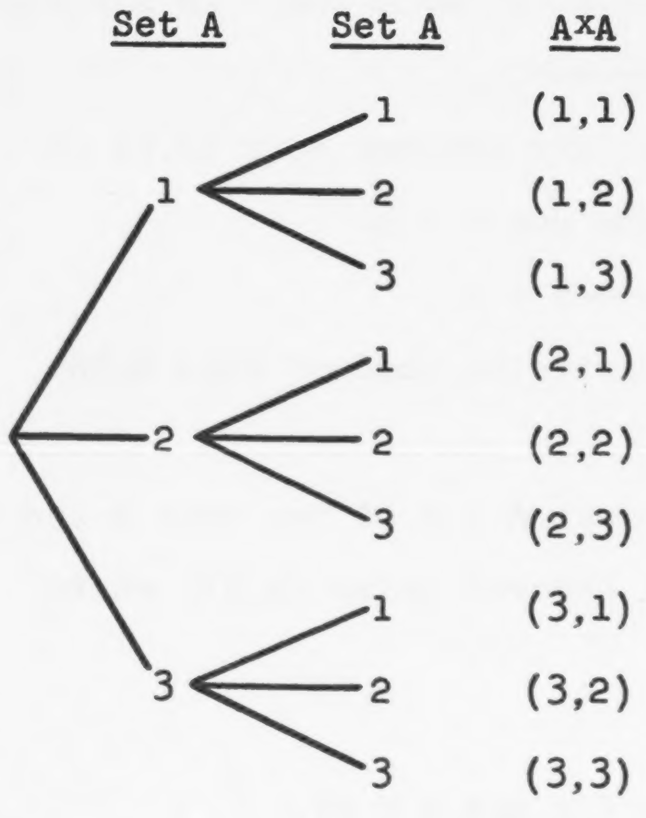
A x A = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3),
(3,1), (3,2), (3,3)}

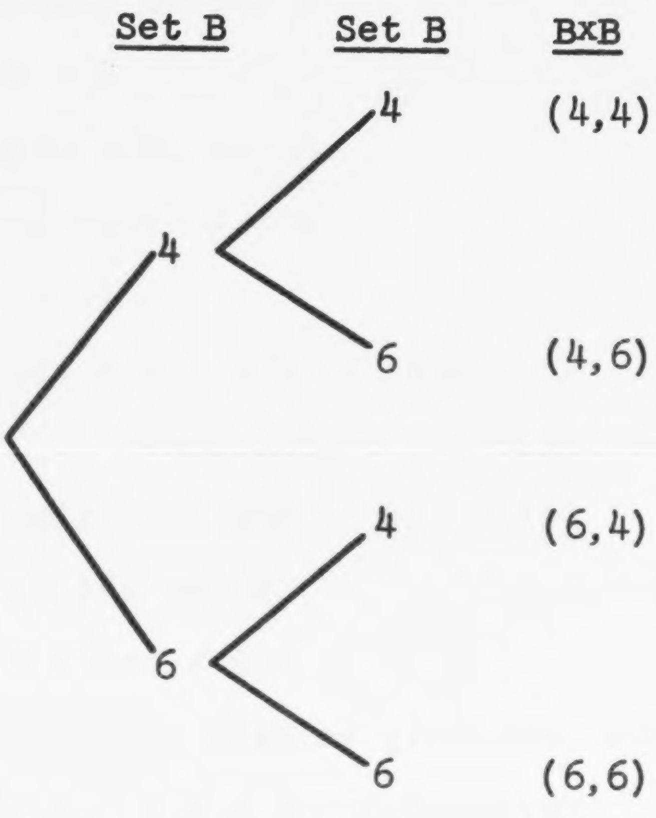
B x B = {(4,4), (4,6), (6,4), (6,6)}

We can also use tree diagrams to represent Cartesian products. Thus we would have:


Set A
Set B
AxB
1
2
3
4
6
4
6
4
6
(1,4)
(1,6)
(2,4)
(2,6)
(3,4)
(3,6)


Set B
Set A
BxA
4
6
1
2
3
1
2
3
(4,1)
(4,2)
(4,3)
(6,1)
(6,2)
(6,3)


Set A
Set A
AxA
1
2
3
1
2
3
1
2
3
1
2
3
(1,1)
(1,2)
(1,3)
(2,1)
(2,2)
(2,3)
(3,1)
(3,2)
(3,3)


Set B
Set B
BxB
4
6
4
6
4
6
(4,4)
(4,6)
(6,4)
(6,6)

The following examples illustrate other instances where we consider the Cartesian product of a set with itself.

Example 1: Let the set S represent the outcome set of a toss of a single die, that is, S = {1,2,3,4,5,6}. Then S x S would represent the outcome set for the toss of a pair of dice.

Example 2: Let $T = \{5\}$. Then $T \times T = \{(5,5)\}$. Note that $T \neq T \times T$.

Example 3: Let Z represent the set of integers. Then $Z \times Z$ can be represented by the set of lattice points in the plane.

Example 4: Let W be the set of whole numbers. The operation of addition on W, denoted by "+," is a mapping which assigns to every element of $W \times W$ a unique element of W called a sum. In symbols

$$W \times W \xrightarrow{+} W.$$

Under this mapping, any ordered pair (a,b) in $W \times W$, maps into the sum $a + b$.

$$(a,b) \xrightarrow{+} a + b$$

We summarize our ideas about Cartesian product sets with the following definitions:

Definition: The Cartesian product $A \times B$ of two sets A and B is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$.

More compactly we have

Definition: $A \times B = \{(a,b): a \in A \text{ and } b \in B\}$.

Now consider the set

$$S = \{-5,-4,-3,-2,-1,0,1,2,3,4,5\}.$$

The product set consists of all possible ordered pairs of these elements

$$S \times S = \{(x,y): x \in S \text{ and } y \in S\}$$

or

$S \times S = \{(-5,-5), (-5,-4), --- (-4,-5), (-4,-4), (-4,-3) ---$
$(0,-5), (0,-1) --- (5,5)\}$

Let us select those pairs of the product set in which the first number x, is *related* to the second number y by the expression

x is a square root of y

By testing each pair we find the subset

$R = \{(-2,4), (-1,1), (0,0), (1,1), (2,4)\}$

We call this subset of a product a relation on S. The elements of S are related by the expression "is a square root of."

In a similar way consider the set

$$A = \{1,2,3,4\}.$$

The product set is

$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4) ---$
$(4,3), (4,4)\}$

Let us select those elements of A x A for which the first element a is related to the second member y by the expression

a is less than b or $a < b$.

It is easy to find that this subset of A x A is the set

$$Y = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}.$$

Thus $a < b$ is a relation on the set A and it is a subset of A x A.

Another relation could be the subset for which $a = b$. Then

$$X = \{(1,1), (2,2), (3,3), (4,4)\},$$

or

$$X = \{(a,b): a \in A, b \in A, \text{ and } a = b\}.$$

The relation $a = b - 2$ would be the subset $M = \{(1,3), (2,4)\}$. Generally, any subset of a Cartesian product is a relation. Thus for A x A, the following are relations:

$R_1 = \{(2,1), (1,2), (3,1), (1,3), (4,1), (1,4)\}$

$R_2 = \{(2,1), (3,2), (4,3)\}$

$R_3 = \{(1,1), (2,1), (3,3), (4,2)\}$

In R_3 the four elements were picked out at random. However, most useful relations are given by some explicit relational phrase.

To determine the relation we may do the following:

(1) Select a relational phrase.

(2) Flank the phrase on the left by the first element and on the right by the second element of an ordered pair. Do this for every element of the Cartesian product.

(3) Determine the truth or falsity of the resulting statement.

(4) The subset of ordered pairs that yield true statements is a relation.

As another example again let

$$S = \{-5,-4,-3,-2,-1,0,1,2,3,4,5\}.$$

The relational phrase connecting the elements of (x,y) is

x is the square of y.

Some instances are

(-5,0): -5 is the square of 0 (False)

(0,0): 0 is the square of 0 (True)

(4,2): 4 is the square of 2 (True)

(4,-2): 4 is the square of -2 (True)

(2,1): 2 is the square of 1 (False), etc.

The set of ordered pairs for which we achieve true statements is

$$R = \{(0,0), (1,1), (1,-1), (4,2), (4,-2)\}.$$

We can speak of this as the "square of" relation and write $x = y^2$.

If an ordered pair of elements (a,b) is in the relation R then we shall express this by writing

$$(a,b) \in R$$

or by writing

$$aRb.$$

We read this latter notation as "a is in the relation R to b." Thus for the relation X we have $(1,1) \in X$, $(2,2) \in X$, $(3,3) \in X$. or equivalently, 1X1, 2X2, 3X3, which are read "1 is in the relation X to 1," etc.

Similarly for the relation Y we have $(1,3) \in Y$ or, equivalently, 1Y3. It may appear strange, at first, to see such statements as "1Y3," However, a familiar example of the "aRb" notation is seen when we consider the relation "equality," denoted by the symbol "=," on the set W x W. If we write "a = b," we mean that "a" and "b" are different names for the same whole number. Thus we may have $1 = 1$, $2 = 1 + 1$, $3 = 2 + 1$, $0 = 1 - 1$, etc. The sentence "a = b" singles out all those ordered pairs in W x W whose first coordinate is the same as the second coordinate, i.e., $R = \{(0,0), (1,1), (2,2),\ldots\}$. The subset of all these ordered pairs therefore defines the "equality" relation on W, and we use the symbol "=" to denote this equality relation. Instead of writing "aRb" we write "a = b."

We consider a few more examples of relations.

Example 1: Let $D = \{2,3,5\}$. We define a relation L on D as follows: $(a,b) \in L$ or $a\ L\ b$ if and only if $a \in D$, $b \in D$, and $a < b$. Thus $L = \{(2,3), (2,5), (3,5)\}$. We could write

$(2,3) \in L$, $(2,5) \in L$, and $(3,5) \in L$ or equivalently

$2 \; L \; 3$, $2 \; L \; 5$, and $3 \; L \; 5$.

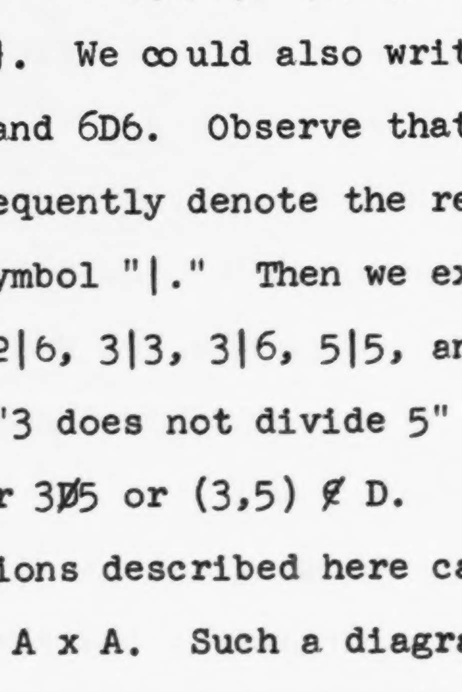
(We usually express the above by writing $2 < 3$, $2 < 5$, and $3 < 5$.)

Example 2: Let $A = \{2,3,5,6\}$. We define a relation D on A as follows: aDb if and only if $a \in A$, $b \in A$, and a "divides" b.

Hence, $D = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6)\}$. We could also write 2D2, 2D6, 3D3, 3D6, 5D5, and 6D6. Observe that $D \subset (A \times A)$.

Note: We frequently denote the relation "divides" by the symbol "|." Then we express the above by $2|2$, $2|6$, $3|3$, $3|6$, $5|5$, and $6|6$. The fact that "3 does not divide 5" could be written as $3 \nmid 5$ or $3 \not{D} 5$ or $(3,5) \notin D$.

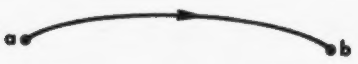
Each of the relations described here can be pictured on a coordinate diagram for A x A. Such a diagram is called a Cartesian Graph. Thus Example 2 can be pictured as follows:



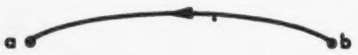
Each lattice point (marked by circles) represents an element of A x A, but only those marked "x" belong to the relation D.

Another convenient way of studying some relations is through the use of arrow diagrams. This device is also called a graph of the relation.

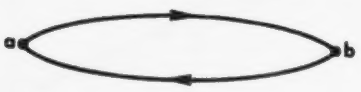
If aRb, then we designate two points and label them "a" and "b." Because aRb we direct an arrow from the point labeled "a" to the point labeled "b."



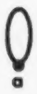
Note that if bRa then the direction of the arrow is reversed.



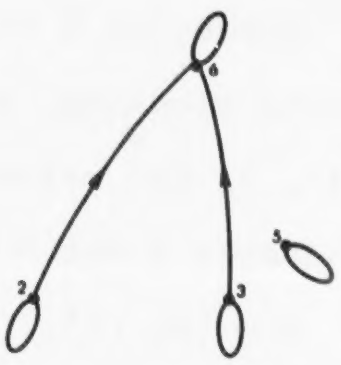
If we have both aRb and bRa then indicate both by:



If it is the case that aRa then we draw a loop at the point labeled "a."



Thus we can draw the following arrow diagram to represent the relation D in Example 2 above:

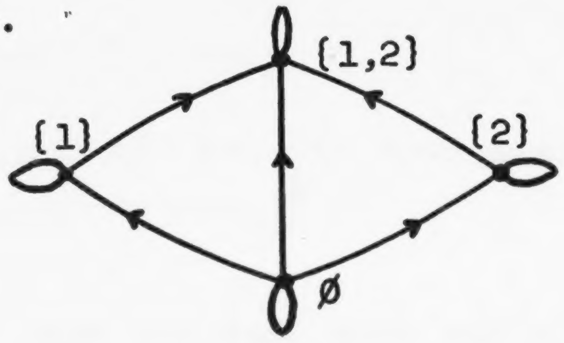


Observe that an arrow is drawn which connects "2" to "6" because 2|6 and also an arrow is drawn which connects "3" to "6" because 3|6. Note that no arrow joins "2" to "5" because 2 does not divide 5 (i.e., $2 \nmid 5$). Note also the loops at each point which indicates that each of the numbers divides itself.

Example 3: Let P be the set of all subsets of the set $\{1,2\}$. The set P then is given by

$$P = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

Consider the relation "is a subset of" denoted by "$\subset$" on the set P. We use an arrow diagram to indicate which elements of P are subsets of each other.



Observe that at each point representing an element of P we have a loop. This is because the elements of P, namely $\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$ are sets, and every set is a subset of itself. Also, "$\emptyset$" is connected to "$\{1\}$," "$\{2\}$," and "$\{1,2\}$" because the empty set $\emptyset$ is a subset of every set. Further, both "$\{1\}$" and "$\{2\}$" are connected to "$\{1,2\}$" since $\{1\} \subset \{1,2\}$ and $\{2\} \subset \{1,2\}$. Do you see that there are nine elements, that is, nine sets of ordered pairs of elements, in the relation "$\subset$" on P?

Example 4: As in Example 3 let P be given by

$$P = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

The relation "is a proper subset of," on the set P is a subset of the relation represented in the arrow diagram above. If the loops are removed from that diagram we have a representation for "is a proper subset of " on P.

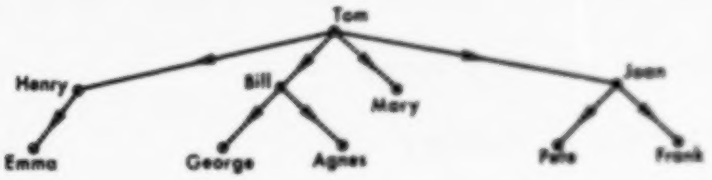
Example 5: Let Z represent the set of integers. Define the relation S on Z as follows: aSb if and only if b is the square of a. Thus

$$S = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), (3,9), (-3,9),\ldots\}.$$

Observe that $S \subset (Z \times Z)$.

Example 6: Let C represent the students in a classroom. Define the relation L on C as follows. Two students x and y are in the relation L on C if and only if x lives within 1 block of y. Can relation L on C be an empty set?

Example 7. The following arrow diagram shows a simplified family tree.



The above tree indicates that Tom had four children, namely Henry, Bill, Mary and Joan. Henry had one daughter, Emma. Bill and Joan each had two children whereas Mary had none. Using the first letters of their names to represent people we see that the relation "is a grandfather of" is the set $\{(T,E), (T,G), (T,A), (T,P), (T,F)\}$.

Let us now summarize some ideas associated with the concept of relation. A binary relation (or relation) R from a set A to a set B is a subset of A x B. If R is a relation from set A to set B, then R assigns to each ordered pair (a,b) in A x B exactly one of the following statements:

i) "a is related to b," written "aRb."

ii) "a is not related to b," written "$a \not R b$."

Since a relation R from a set A to a set B is a subset of A x B, we see that every relation is a set of ordered pairs. We shall be concerned most often with a relation R from a set A to the same set A. We say, in this case, that R is a relation on the set A. Here, of course, $R \subset (A \times A)$.

8.10 Exercises

1. Using Example 7 in 8.9, list the elements in the following relations: (Note: Represent each person by the first letter of his name.)

   (a) is a father of   (d) is an uncle of

   (b) is a brother of   (e) is a sister of

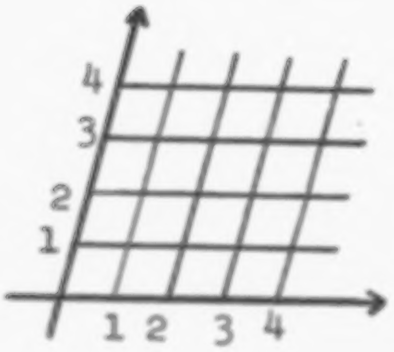
   (c) is a grandmother of

2. Let $P = \{1,2\}$ and $Q = \{2,3,4\}$. Determine the following Cartesian products:

   (a) P x Q   (c) P x P

   (b) Q x P   (d) Q x Q

3. Copy the coordinate scheme given below on your paper. Using Exercise 2 above, graph the following Cartesian products using the symbols indicated:



(a) graph $P \times Q$ using crosses (x)

(b) graph $Q \times P$ using circles (o)

(c) graph $P \times P$ using triangles (Δ)

(d) Determine the following:

   (1) $(P \times Q) \cap (Q \times P)$

   (2) $(P \times P) \cap (Q \times P)$

   (3) $(P \times P) \cap (P \times Q)$

   (4) $P \times (P \cap Q)$

   (5) $(P \times P) \cup (P \times Q)$

   (6) $P \times (P \cup Q)$

(e) On the basis of your answers to 3 (d) above make one more conjectures about the properties of "x."

4. Let $M = \{1,2\}$, $N = \{2,3\}$, and $P = \{4,5\}$.

(a) Determine the following:

   (1) $(M \times N) \cup (M \times P)$

   (2) $M \times (N \cup P)$

(b) What do you observe?

5. Let $A = \{0,2,4\}$ and $B = \{0,1,2\}$. Let R be the relation "is greater than," denoted by ">," from A to B. i.e., aRb if and only if $a > b$, where $a \in A$ and $b \in B$.

(a) Write R as a set of ordered pairs.

(b) Of what set is R a subset?

(c) Explain why 0R2, or why not.

(d) Explain why 4R3, or why not.

6. Let B = {2,4,5,8,15,45,60}. Let R be the relation "divides," denoted by "|," on the set B, i.e., aRb if and only if a|b.

(a) Write R as a set of ordered pairs.

(b) Of what set is R a subset?

(c) Represent the set R by means of an arrow diagram.

(d) Explain how your diagram does or does not indicate the following:

(1) 2|2

(2) 2|4

(3) 2|8

(4) 2|45

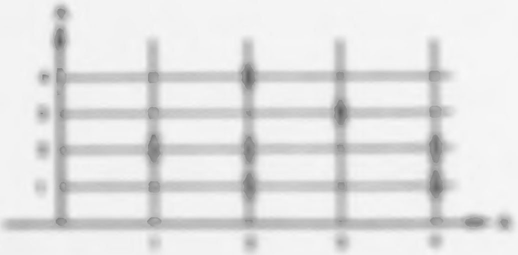
(5) 45|5

(6) 60|60

7. (a) Let S be the set of all subsets of the set {x}. Draw an arrow diagram to represent the relation "is a subset of," denoted by "⊂," on the set S.

(b) Let T be the set of all subsets of the set {x,y,z}. Draw an arrow diagram to represent the relation "is a subset of," denoted by the "⊂," on the set T.

8. Let A = {1,2,3,4}. We define a relation R on A as the set of ordered pairs of numbers designated by crosses (x) in the coordinate diagram of A x A given below.



(a) Explain why each of the following is true or false:

(1) $1R1$

(2) $2R2$

(3) $3R2$

(4) $2R4$

(5) $4 \not R 3$

(6) $4 \not R 2$

(7) $4 \not R 4$

(8) $3 \not R 3$

(b) Find $\{x: (x,2) \in R\}$, that is, find all the elements in A which are related to 2.

(c) Find $\{x: 4Rx\}$, that is, find all the elements in A to which 4 is related.

9. (a) Is every relation a mapping? Explain.

(b) Is every mapping a relation? Explain.

(c) Let the relation R from A to B to sketched on the coordinate diagram of A x B. What test could one devise in order to determine whether or not R is a mapping of A into B?

10. Research Problem: If set A has $\underline{m}$ elements and set B has $\underline{n}$ elements, how many different relations could we define from A to B? Experiment and write a report of your findings.

8.11 Properties of Relations

In this section we shall consider a relation R only if it is a subset of the Cartesian product of some set A with itself. That is,

$$R \subset A \times A$$

Again we shorten this by saying R is a relation on the set A.

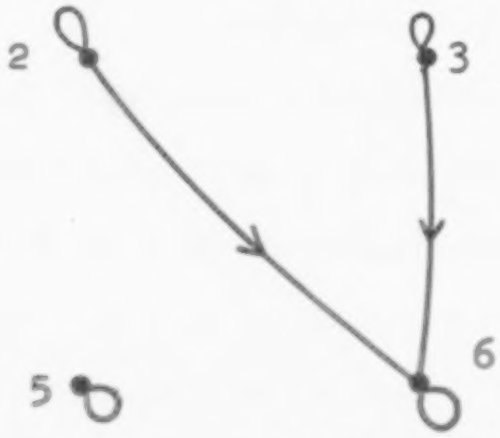
Consider the set

$$A = \{2,3,5,6\}$$

and the relation on it "divides." This relation is the subset of A x A

$$D = \{ (2,2), (2,6), (3,3), (3,6), (5,5), (6,6) \}$$

and illustrated by the following diagram.



This relation D has a particular property indicated by the fact that there is a loop for each element of the set A. Since $2|2$, $3|3$, $5|5$, $6|6$, we see, that for each element a of A

$$aDa \text{ or } (a,a) \in D$$

We describe this property by saying D is a reflexive relation on A.

Similarly, the relation "$\subset$" on the set $P = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$, as given in Example 3 in 8.9 is a reflexive relation on P. Again, the arrow diagram indicates this reflexive property by a loop at each point.

The relation "is a proper subset of" on the same set P, as given in Example 4 in 8.9 is not a reflexive relation on ?, because it is not true that a set is a proper subset of itself.

Let us make a precise statement concerning the property of reflexivity that a relation on a set may or may not possess.

Definition: Let R be a relation on a set A. R is called a _reflexive_ relation on A if and only if, for every $a \in A$, $(a,a) \in R$ or aRa. In other words, R is reflexive on A if and only if every element in A is related to itself.

Question 1: Let S be the relation on Z given in Example in 8.9, that is
$S = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), \ldots\}$.
Is S reflexive on Z? Explain.

Question 2: Let $V = \{1,2,3,4,5\}$. Let R be the relation on V given by $R = \{(1,1), (1,2), (2,2), (2,3), (3,3), (4,4), (5,5)\}$.
Is R reflexive on V? Explain.

Question 3: As in Example 6 of 8.9, let C represent the students in a classroom. Let L denote the relation "lives within 1 block of" on C. Is L reflexive on C? Explain.

Certainly, one of the most basic relations that we encounter is that of "equality," denoted by "=." For example, if W is the set of whole numbers then $x = x$ for all $x \in W$. Hence equality is a reflexive relation on the set W. (In fact equality is reflexive on any set.) Another important property of the relation equality is the following: If x, y are whole numbers and x = y, then y = x. We express this property by saying that equality is a symmetric relation on the set W.

Again let C represent the students in a certain school and L denote the relation "lives within 1 block of" on C. It is evident that if Bill lives within one block of Jim, then Jim lives within one block of Bill. In general, if x lives within one block of y, then y lives within one block of x. The relation "lives within one block of" is a symmetric relation on C.

As another example, suppose that M represents only the male children (boys) in the school and let B denote the relation on M defined by "is a brother of." It is clear that if Bill is a brother of Jim, then Jim is a brother of Bill. In general if $x \in M$ and $y \in M$, then xBy implies yBx. The relation B is a symmetric relation on M.

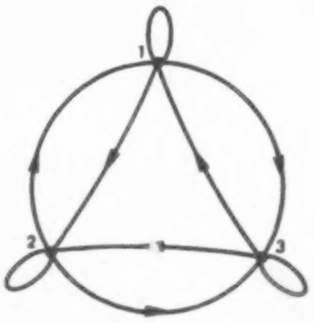
Now suppose that C is the set of all children in the school (including girls as well as boys); then we could have xBy but not yBx (for example suppose Jim is a brother of Jane). Thus the relation"brotherhood" is not symmetric on set C, although it is symmetric on set M. The above examples suggest the following definition:

Definition: Let R be a relation on a set A and let a and b be any elements of A. We say R is a symmetric relation on A if and only if aRb implies bRa.

Example 1: Let K = {1,2,3}. An easily found relation R on K is the Cartesian product KxK. Since KxK $\subset$ KxK, KxK is a relation R on K. We find that

R = {(1,1), (1,2), (1,3), (2,1), (2,2),
(2,3), (3,1), (3,2), (3,3)}.

Note that 1 R 2 implies 2 R 1, 1 R 3 implies 3 R 1, etc. If we consider the arrow diagram of R on K we observe



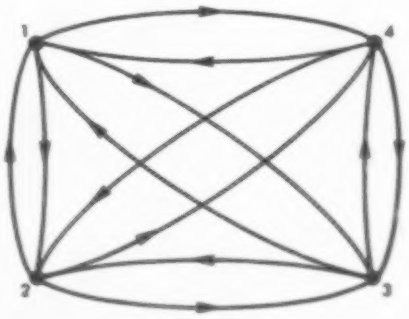
that R is a symmetric relation of K since whenever there is an arrow from a to b there is a corresponding arrow from b to a. Recall that the loops at each point signify R is also a reflexive relation on K.

Example 2: Let K = {1,2,3,4,5} and consider the relation R on K defined by "evens or odds;" that is, if a, b ∈ K, then a R b if and only if a and b are both even whole numbers or both odd whole numbers. This means that

R = {(1,1), (1,3), (1,5), (2,2), (2,4),
(3,1), (3,3), (3,5), (4,2), (4,4),
(5,1), (5,3), (5,5)}.

Notice that (1,3) ∈ R and (3,1) ∈ R since 1 and 3, regardless of the order in which you consider them, are both odd whole numbers in K. Is it always true that whenever (a,b) ∈ R, it follows that $(b,a) \in R$? If so, then we may say that the relation R is symmetric.

Example 3: Let $J = \{1,2,3,4\}$. Let us define a relation S on J as follows: If $a, b \in J$ then aSb if and only if $a \neq b$. Thus 1S4 because $1 \neq 4$. Also 4S1 because $4 \neq 1$. The arrow diagram for S on J indicates that S is a symmetric relation on J.



Example 4: Let Z be the set of integers. The relation "less than or equal to," denoted by "$\leq$" is not a symmetric relation on Z because for all $a, b \in Z$, $a \leq b$ does not imply that $b \leq a$. For example, $3 \leq 4$ does not imply $4 \leq 3$.

The relation described in Example 4, that is "$\leq$" on Z is not symmetric.

The next property that we shall examine is illustrated by the following: We know for the set W of whole numbers that if $a = b$ and $b = c$, then $a = c$. The relation of "equality" is said to be a transitive relation on W. The general property is given in the following definition.

Definition: Let R be a relation on a set A where a, b, and c are any elements of A. We say that R is a transitive relation on A if and only if whenever aRb and bRc, then aRc.

Example 1: Let Z be the set of integers and let R be the relation on Z defined by "x is less than y." Then R is a transitive relation on R since

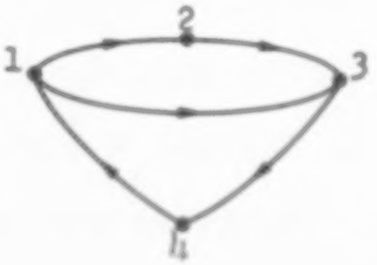
if $x < y$ and $y < z$, then $x < z$.

In particular, we note that not only do we have $0 < 7$ and $7 < 100$, we also have $0 < 100$. Again, not only do we have $-5 < -3$ and $-3 < -1$, but we also have $-5 < -1$.

Example 2: Let H = {1,2,3,4}. Let us define a relation R on H as follows:

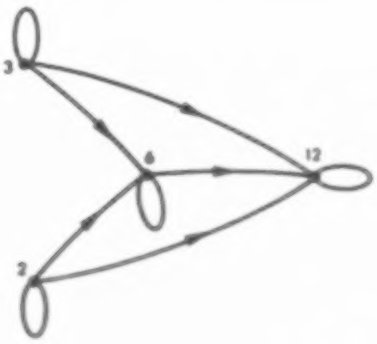
R = {(1,2), (2,3), (1,3), (3,4), (4,1)}.

If we examine the arrow diagram of the relation R on H we see that not only do we have 1R2 and 2R3, but we also have 1R3.



So far so good. But now observe 1R3 and 3R4. Does this imply that 1R4? It does not! Actually the arrow points from 4 to 1. This means 4R1 and indeed 1R4 is not in the relation. Since the transitive property fails for at least one triple of elements of H, we say that R is not transitive on H.

Example 3: Let B = {2,3,6,12} and let D be the relation "divides" on B. The arrow diagram below indicates that $3|6$ and $6|12$. We therefore check to see if $3|12$. The arrow diagram indicates this is indeed so. Next we observe that $2|6$ and $6|12$, and we note that $2|12$.



In the present case, these two verifications are sufficient to show that D is a transitive relation on B. (All the other possibilities are trivial.)

We have pointed out in this section that the important relation of "equality" on the set W satisfies three properties, namely, the reflexive, the symmetric and the transitive properties. That is.

(i) Reflexivity. For every whole number a, $a = a$.

(ii) Symmetry. For any whole numbers a and b whenever $a = b$, then $b = a$.

(iii) Transitivity. For any whole numbers a, b, and c, whenever $a = b$ and $b = c$, then $a = c$.

In the next section we shall see that if a relation on a set has these three properties some important results can be derived. Any relation on a set which has these properties is called an *equivalence* relation. Thus we have the following

Definition: A relation R on a set A is an *equivalence relation* if and only if

(1) R is reflexive; that is, for every $a \in A$, aRa.

(2) R is symmetric; that is, for every *a* and *b* in A, whenever aRb, then bRa.

(3) R is transitive; that is, for every *a*, *b*, and *c* in A, whenever aRb and bRc, then aRc.

Example 1: Consider the relation defined by "has the same first name as" on the set C of students in a classroom. We check to see that the requirements in the above definition are satisfied. Let *x*, *y* and *z* be any students in the class. Then

(i) *x* has the same first name as *x*;

(ii) if *x* has the same first name as *y*, then *y* has the same first name as *x*;

(iii) if *x* has the same first name as *y* and *y* has the same first name as *z*, then *x* has the same first name as *z*.

Since each of the above is true, the relation defined by "has the same first name as" is (i) reflexive, (ii) symmetric, and (iii) transitive on C, and hence is an equivalence relation on C.

Example 2: Consider the relation "$\subset$" on all the subsets of $A = \{a,b\}$. We find that "$\subset$" is reflexive and transitive on A, but although $\{a\} \subset \{a,b\}$ it is not true that $\{a,b\} \subset \{a\}$. We see that the relation is not symmetric on A. Hence it is not an equivalence relation on A.

Example 3: Let P be the set of all people in the United States, and let T be the relation "has the same blood-type as." (We say "aTb" if and only if, *a* has the same blood-type as *b*.) Is T an equivalence relation on P? Yes, because

(1) T is reflexive on P. (Everyone has the same blood-type as himself.)

(2) T is symmetric on P. (Whenever *a* has the same blood-type as *b*, then *b* has the same blood-type as *a*.)

(3) T is transitive on P. (Whenever *a* has the same blood-type as *b*, and *b* has the same blood-type as *c*, then *a* has the same blood-type as *c*.)

Example 4: Let B be the set of all savings banks in the United States, and let D be the relation on B defined by "pays the same interest rate as." We readily verify that D is reflexive, symmetric and transitive on B, and hence D is an equivalence relation on B.

Example 5: Let T be the set of maximum temperatures recorded for each day of the year 1966. (T is therefore represented by a list of 365 temperature readings.) Let R be the relation on set T defined by: xRy if and only if temperature *x* differs from temperature *y* by less than two degrees. Is R an equivalence relation on the set T? Let us check each requirement.

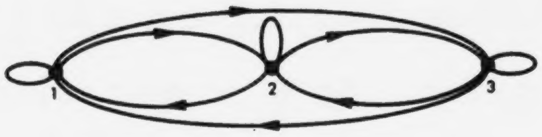
(1) Is R reflexive? Yes, because each temperature reading certainly differs from itself by less than two degrees.

(2) Is R symmetric? Yes, because whenever temperature *x* differs from temperature *y* by less than two degrees then temperature *y* certainly differs from temperature *x* by less than two degrees.

(3) Is R transitive? No, because if temperature *x* differs from temperature *y* by less than two degrees and temperature *y* differs from temperature *z* by less than two degrees, it does not follow that temperature *x* must differ from temperature *z* by less than two degrees. (Suppose for example that *x* is 1½ degrees higher than *y* and *y* is 1½ degrees higher than *z*. Then *x* is 3 degrees higher than *z*!) Thus, although the relation R is both reflexive and symmetric,

it is not transitive and hence is not an equivalence relation on the set T.

Example 6: Let K = {1,2,3} with a relation defined on it which is illustrated by the arrow diagram below.



Examine this diagram and convince yourself that the relation illustrated is (i) reflexive, (ii) symmetric, and (iii) transitive on K, and hence is an equivalence relation.

8.12 Exercises

1. Let E = {1,2,3} with the following relation R defined on it.
   R = {(1,1), (1,2), (2,3), (2,2), (3,3), (2,1)}
   (a) Explain why R is a relation on E.
   (b) Draw an arrow diagram which represents R on E.
   (c) Explain why R is or is not (1) reflexive, (2) symmetric, (3) transitive
2. Let S be a relation on a set F, where F = {1,2,3,4} and
   S = {(1,1), (1,3), (2,2), (2,3), (2,1), (3,2), (3,3), (3,4), (4,1)}.
   (a) Draw an arrow diagram for S.
   (b) Explain why S is or is not (1) reflexive, (2) symmetric, (3) transitive.

3. Each of the following open sentences defines a relation on the set W of whole numbers. Determine for each if it is or is not a reflexive relation on W.
   (a) *a* is less than or equal to *b*.
   (b) $a = 8 - b$.
   (c) *a* divides *b*.
   (d) *a* is greater than *b*.
   (e) *a* is equal to *b*.
   (f) the square of *a* is *b*.
   (g) $a - b$ is divisible by 5.
4. In Exercise 3, determine which relations are symmetric on W.
5. In Exercises 3, determine which relations are transitive on W.
6. Which of the relations in Exercise 3, if any, are equivalence relations.
7. (a) When is a relation R on a set A not reflexive?
   (b) When is a relation R on a set A not symmetric?
   (c) When is a relation R on a set A not transitive?
8. Let $A = \{1,2,3\}$. Consider the following relations on A:

   $R_1 = \{(1,1), (1,2), (1,3), (2,1), (2,3)\}$

   $R_2 = \{(1,1), (2,3), (3,2), (1,2), (3,1)\}$

   $R_3 = \{(1,2), (2,3), (1,3)\}$

   $R_4 = \{(1,1)\}$

   $R_5 = A \times A$

   Determine which of these relations is, and which is not
   (a) reflexive
   (b) symmetric
   (c) transitive.

9. Examine the relation defined by "is a brother of" for a set of people with respect to

   (a) reflexivity,

   (b) symmetry,

   (c) transitivity.

10. Let $A = \{2,4,6\}$. Consider the following relations on A:

    $R_1 = \{(2,2), (4,2), (4,4), (4,6)\}$

    $R_2 = \{(2,2), (4,6), (6,4)\}$

    $R_3 = A \times A$

    $R_4 = \{(2,2)\}$

    $R_5 = \{(2,4)\}$

    (a) Determine which of these relations is (1) reflexive (2) symmetric (3) transitive, on the set A.

    (b) Indicate which, if any, are equivalence relations on A.

11. Let L be a set of lines in the plane and let P be the relation on L defined by "ℓ_1 is parallel to ℓ_2." Determine whether or not P is (a) reflexive, (b) symmetric, (c) transitive, (d) an equivalence relation. (Assume a line is parallel to itself.)

12. Let S be the collection of subsets of $\{x,y,z\}$. If A and B are elements of S the following describe relations on S:

    (i) "$A \subset B$"

    (ii) "$A \subset B$ and A is not equal to B"

    (iii) "A is disjoint from B"

    Determine which of the above relations on S are (a) reflexive (b) symmetric, (c) transitive.

*13. A relation R on a set A is called irreflexive if and only if $a \not R a$, for all $a \in A$.

(a) Which of the relations in Exercise 3 above are irreflexive?

(b) Which of the relations in Exercise 8 above are irreflexive?

(c) Which of the relations in Exercise 10 above are irreflexive?

(d) Which of the relations in Exercise 12 above are irreflexive?

*14. Definition: Let R be a relation on a set A where a and b are any elements of A. We say R is an anti-symmetric relation on A if and only if whenever aRb and bRa, then a = b.

(a) Which of the relations in Exercise 3 above are anti-symmetric?

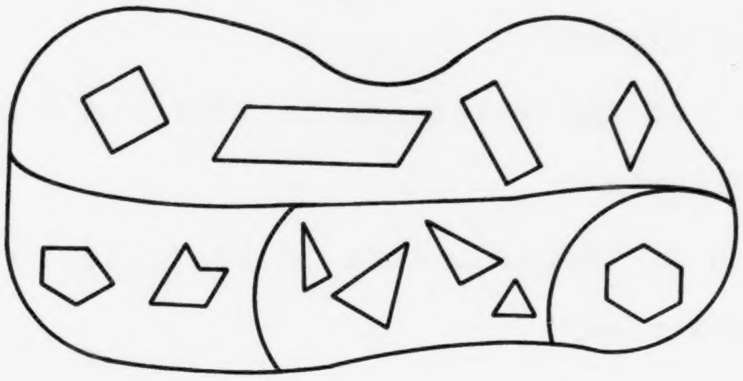
(b) Which of the relations in Exercise 8 above are anti-symmetric?

(c) Which of the relations in Exercise 10 above are anti-symmetric?

(d) Which of the relations in Exercise 12 above are anti-symmetric?

8.13 Equivalence Classes and Partitions

Examine the drawing below in which we have drawn a closed curve about a set of eleven geometric figures. Designate this set of figures as G.



Not all of the figures have the same number of sides. In fact there are four 3-sided figures (i.e., 4 triangles), four 4-sided figures (i.e., 4 quadrilaterals), two 5-sided figures (i.e. 2 pentagons) and one 6-sided figure (i.e., 1 hexagon).

We define a relation R on the set G as follows: If x and y are any elements of G we say that xRy if and only if x and y have the same number of sides.

Thus any two triangles in G are in the relation R to each other whereas a triangle and a quadrilateral are not in the relation R to each other.

Because every geometric figure in G has the same number of sides as itself, we see that R is reflexive on G. If x has the same number of sides as y, then y has the same number of sides as x. Hence, R is symmetric on G. Also if x has the same number of sides as y and y has the same number of sides as z, then x has the same number of sides as z. Thus R is transitive on G. From the above we conclude that R is an equivalence relation on G.

Now examine the effect of the equivalence relation R on the set G. It is important to note that the relation R effects a separation of the elements of G into disjoint subsets. Each of these subsets contains exactly those geometric figures which have the same number of sides. (See how this is indicated in the drawing above.) Designate these subsets of G as T (the set of triangles), Q (the set of quadrilaterals), P (the set of pentagons), and H (the set of hexagons). The collection of subsets of G

$$\{T, Q, P, H\}$$

produced by the equivalence relation R on G is called a partition of G.

The subsets which form the partition of G have two important properties:

Property 1: The union of the subsets T, Q, P, and H of G is the set G. That is $T \cup Q \cup P \cup H = G$

Property 2: The subsets T, Q, P, and H of G are pairwise disjoint. This means that the intersection of any two distinct subsets is the empty set. This is true because a geometric figure cannot be both a triangle and a quadrilateral. Hence $T \cap Q = \emptyset$. Similarly $T \cap P = \emptyset$, $T \cap H = \emptyset$, $Q \cap H = \emptyset$, and $P \cap H = \emptyset$.

It is no accident that R effected a partition of G into pairwise disjoint subsets whose union is G. We obtained such a partition of G because R is an equivalence relation on G. The most significant property of an equivalence relation on a set is that it always partitions the set into pairwise disjoint subsets whose union is the given set.

We could also say that an equivalence relation R on a nonempty set A partitions the set by putting those elements which are related to each other in the same subset of A. Each of these subsets is called an *equivalence class*. In the example above T, Q, P, and H are equivalence classes. The following examples will illustrate many of the ideas examined above.

Example 1: Let us define a relation R on the set Z of integers as follows: Let *x* and *y* be any two integers. We say xRy if and only if both *x* and *y* are even or both *x* and *y* are odd. Note: Zero is even. Thus -3R7 but $-3\not R 8$. The relation R is an equivalence relation on Z. (Prove this.) Moreover the relation R establishes two subsets of Z:

$E = \{x: x \in Z, \text{ and } x \text{ is even}\}$ and

$O = \{x: x \in Z, \text{ and } x \text{ is odd}\}$.

Every integer in Z is either an element in E or an element in O, but never an element in both E and O.

(i) $E \cup O = Z$, and

(ii) $E \cap O = \emptyset$.

The equivalence relation R on Z effects a partition on Z. This partition is $\{E, O\}$. E and O are equivalence classes in this partition.

Example 2: Let $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$. We define a relation R on A as follows: Let *a* and *b* be any elements of A. We say aRb if and only if *a* and *b* have the same remainder when they are divided by 4. It is easy to see that R is an equivalence relation on A which determines the following subsets of A:

$B_1 = \{1,5,9\}$, $B_2 = \{2,6,10\}$, $B_3 = \{3,7,11\}$,

$B_4 = \{4,8,12\}$.

We note that $B_1 \cup B_2 \cup B_3 \cup B_4 = A$ and also that $B_1 \cap B_2 = \emptyset$, $B_1 \cap B_3 = \emptyset$, $B_1 \cap B_4 = \emptyset$, $B_2 \cap B_3 = \emptyset$, $B_2 \cap B_4 = \emptyset$, $B_3 \cap B_4 = \emptyset$.

Thus R effects the partition $\{B_1, B_2, B_3, B_4\}$ on A.

Example 3: Let C be the set of students in a class. It is clear that the relation "has the same first name as" is an equivalence relation on C. Further, this relation partitions C into equivalence classes. (Examine your own class.) It could happen that every student had a different name. If in such a class there are twenty students we find that the equivalence relation still partitions the set. Here each equivalence class would have in it a single element. Thus, the partition would be a set having twenty equivalence classes as elements.

Example 4: Let $A = \{0,1,2\}$. We find that there are five different possible partitions of A. These are,

(i) $\{ \{0,1,2\} \}$

(ii) $\{ \{0\}, \{1,2\} \}$

(iii) $\{ \{1\}, \{0,2\} \}$

(iv) $\{2\}, \{0,1\} \}$

(v) $\{0\}, \{1\}, \{2\} \}$

Each of the five sets above is a partition of A. Again we see that the elements of a partition are sets. In (ii) the elements that make up the partition $\{ \{0\}, \{1,2\} \}$ are the equivalence classes $\{0\}$ and $\{1,2\}$. We observe that $\{0\} \cup \{1,2\} = \{0,1,2\} = A$. Also $\{0\} \cap \{1,2\} = \emptyset$. Similar statements are possible for (i), (iii), (iv), and (v).

Example 5: Let P be set of all people in the United States and let T be the relation on P defined by "has the same type blood as." We saw in Section 8.13 that T is an equivalence relation on P. Hence T partitions P into equivalence classes. These equivalence classes are called "blood groups."

We are seldom interested in a set unless some relation or operation has been defined on the set. In this section we have seen that defining an equivalence relation R on a set A yields a partition of A into equivalence classes. We might say that the relation R on A gives a "structure" to the set A. Of course different relations defined on A yield different structures. We shall encounter many ways of structuring sets. The study of structures on sets is considered by some mathematicians to be the very essence of mathematics itself.

8.14 Exercises

1. Let $A = \{1,2,3,4,5,6\}$. Explain why each of the following is or is not a partition of A.

(a) { {1,2}, {5,6,3} }

(b) { {1,2}, {3}, {4,5}, {6,2} }

(c) { {1,3,5}, {2,4,6} }

(d) { {1}, {2}, {3}, {4}, {5}, {6} }

(e) { {1}, {6,4}, {3,5,2} }

(f) { {1,2,3,4}, {4,5,6} }

(g) { {1,2,3,4,5,6} }

(h) { {1,2}, {3,4} }

2. Find all the partitions of {1,2}.

3. Explain why "<" defined on W does not partition W.

4. Let R be an equivalence relation on A. If we assume that cRa and cRb, why can we conclude that aRb?

5. Find all the partitions of {1,2,3,4}.

6. Let S be the set of all lattice points in the plane and let R_1 and R_2 be relations on S defined as follows:

R_1: "has the same first coordinate as"

R_2: "has the same second coordinate as"

(a) Show that both R_1 and R_2 are equivalence relations on S.

(b) Describe the equivalence classes into which S is partitioned by each of the relations R_1 and R_2.

7. Consider the following relations defined on the set P of people in the United States:

R_1: "lives in the same state as"

R_2: "lives within 1 mile of"

R_3: "is the father of"

R_4: "is a member of the same political party as"

R_5: "has the same I.Q. as"

(a) Determine which of the above are equivalence relations on P.

(b) Describe the equivalence classes in the partititions effected by the relations in (a) which are equivalence relations on P.

*8. Research Problem: Let R be an equivalence relation on A. For every $a \in A$, let

$$B_a = \{x: xRa\}$$

Prove that these sets B_a are the equivalence classes in the partition of A effected by R.

*9. A partial ordering of a set A is a relation on A which is

(1) reflexive, i.e., for every $a \in A$, aRa

(2) anti-symmetric, i.e, for all a, $b \in R$, aRb and bRa implies $a = b$.
(See Exercise 14 in Section 8.14.)

(3) transitive, i.e., for all a, b, $c \in R$, aRb and bRc implies aRc.

  (a) Let S be the collection of all subsets of $\{1,2\}$. Show that the relation "$\subset$" defined on S is a partial ordering of S.

  (b) Draw the arrow diagram for this relation. Try to describe a general property which the arrow diagram for any partial ordering relation must have.

*10. (a) Use an arrow diagram to illustrate the relation "divides" on the set $E = \{1,2,3,4,5,6\}$. Is this relation a partial ordering on E?

(b) Explain why the relation "<" is or is not a partial ordering of the set W.

8.15 Summary

In this chapter you have encountered some of the most basic terms used in the study of mathematics. Terms such as set, relation, equivalence class, partition, etc. will become part of your basic vocabulary.

With respect to sets you should be able to give a clear and complete description of the following terms:

set equality, subset, proper subset, universal set, union, intersection, empty set, complement, disjoint sets, Cartesian product set.

With respect to relations you should understand what is meant by the following terms:

relation, reflexivity, symmetry, transitivity, equivalence relation, partition.

Also you should be aware of the tools we have used in our study. These tools include:

set notation, Venn diagrams, arrow diagrams.

At this time you should review for yourself the meanings of the above terms. Restudy any terms whose meanings are not clear to you.

8.16 Review Exercises

1. Let S be a universal set where $S = \{-3,-2,-1,0,1,2,3\}$.

Let $A = \{-3,-2,-1,0\}$, $B = \{1,2,3\}$, $C = \{-3,-1,1,3\}$, $D = \{0\}$.

(a) Determine the following sets:

(1) $A \cup B$ (5) $B \cup C$ (9) $D \cup D$

(2) $A \cap B$ (6) $B \cap C$ (10) $B \cap B$
(3) $A \cup C$ (7) $A \cup D$ (11) $S \cap D$
(4) $A \cap C$ (8) $A \cap D$ (12) $S \cup D$

(b) Find the complement of each of the following sets:

(1) A (3) C (5) $A \cup D$
(2) B (4) $A \cap C$ (6) $A \cup B$

(c) Which of the sets A, B, C, D are

(1) subsets of the other?

(2) proper subsets of the other?

(3) pairwise disjoint?

2. Write three statements that are true of each set A.

3. Let $B = \{x\colon x \in W \text{ and } x \text{ is even and } x < 3\}$.

(a) Rewrite set B by listing its elements.

(b) List all the subsets of B.

(c) List all the proper non-empty subsets of B.

(d) Determine $B \times B$.

(e) Is $\{(0,0), (0,1)\}$ a relation on B?

(f) Is $\{(0,0), (0,2)\}$ a relation on B?

(g) Draw an arrow diagram for $B \times B$.

4. Let $V = \{0,1,2,3\}$. Let R be a relation on V defined as follows:

$R = \{(0,0), (0,1), (1,0), (1,2), (2,1), (2,2), (0,2), (3,3)\}$

(a) Draw the arrow diagram for R on V.

(b) Is R an equivalence relation on V?

(c) What would occur if we defined a new relation S on V where

$$S = R \cup \{(1,1), (2,0)\}?$$

5. (a) Prove or disprove that $(A \cap B) \cup (A \cap \overline{B}) = A$

   (b) $(A \cap B) \cup (A \cap \overline{B}) \cup (A \cap B) = ?$

6. Give an example of a relation R on a set A which is,

   (a) reflexive and transitive, but not symmetric.

   (b) reflexive, symmetric, and transitive

   (c) neither reflexive, nor symmetric, nor transitive.

   (d) transitive but neither reflexive nor symmetric.

   (e) symmetric and transitive but not reflexive.

7. Determine which of the following are true:

   (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

   (b) If $A \subset B$, then $\overline{B} \subset \overline{A}$.

8. Let $D = \{2,4,6,8,10,12\}$.

   Explain why the following are or are not partitions of D:

   (a) $\{ \{2,4\}, \{6,10\}, \{4,12\}, \{8\} \}$

   (b) $\{ \{2,4,6\}, \{8,10\}, \{12\} \}$

   (c) $\{ \{2\}, \{6,12\}, \{4,10\} \}$

9. Let $B \circ C = B \cap \overline{C}$. Prove or disprove that

$$A \cap (B \circ C) = (A \cap B) \circ (A \cap C).$$

CHAPTER 9

TRANSFORMATIONS OF THE PLANE

9.1 Knowing How and Doing

Have you ever read a book on how to roller skate or ride a bicycle? Do you think you could have done well on roller skates or on a bicycle the very first time you tried merely because you have read the book? Knowing how is not quite the same as being able.

In this chapter you will be given a chance to do many things as well as to learn about them. In order to do these things you will need some equipment in addition to pencil and paper. At the beginning of each section you will be told what equipment you will need. Obtain this equipment before going further so that you can read and follow without interruptions.

9.2 Reflections in a Line

Materials needed: Paper without lines, tracing paper, ink, pen, two small rectangular mirrors, and a compass.

Activity 1: Fold one unlined sheet of paper down the middle. Open up this folded sheet and put one drop of ink in the crease and one drop of ink about an inch away from the crease.

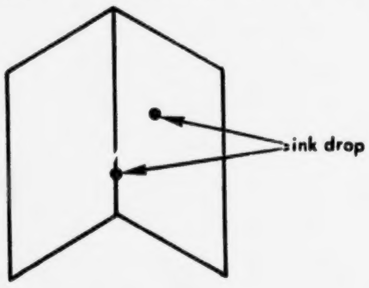


Figure 9.1

Close the paper carefully to spread the ink about, keeping the ink within the folded paper.

Now open up the paper. Look at the ink spots on both paper halves. How do the ink spots compare in size and shape?

Now fold one half back and replace it by a mirror in an upright position so that the edge of the mirror fits into the crease. How do the images you see in the mirror compare with the ink spots you folded back?

Put 2 more ink drops on one half of your paper and repeat the steps of the preceding paragraph. Compare the distance between any 2 ink spots on one paper half with the distance for the corresponding 2 ink spots on the other. Are they the same? What generalization seems to hold for the two paper halves? The ink spot figure on one paper half is called the reflection in the crease of the other ink spot figure.

After the ink dries, place a piece of tracing paper carefully over the ink figured sheet. Then trace one of the ink spot figures. What must you do to the tracing paper to get a picture of the reflection of the figure you traced? Test your solution with another ink spot figure.

In previous chapters we have learned that a mapping makes assignments. For example, the successor mapping, S, assigns to each integer the next larger integer.

$$n \xrightarrow{S} n + 1$$

A reflection in a line is also a mapping since it assigns points to points on a plane. Restricting ourselves to a fixed plane, a reflection with respect to a fixed line assigns to each point its mirror image or reflection in the given line. The next activity suggests important properties of reflection mappings.

Activity 2: Fold an unlined sheet of paper down the middle. Open up the folded sheet and place a heavy dot off the crease line; label the dot "A."

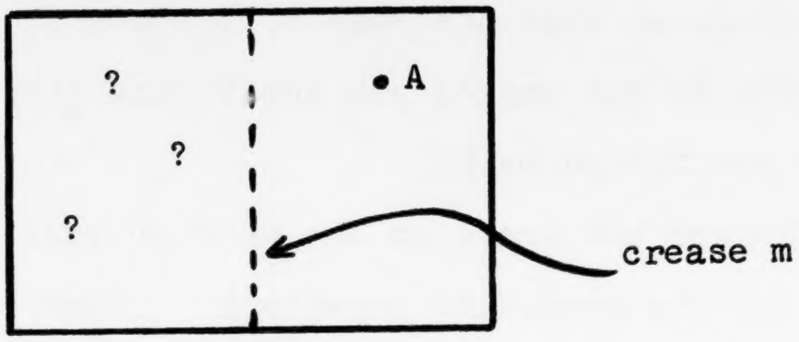


Figure 9.2

Where do you think the reflection of A in m will be?

Now close the paper again folding from left to right with the paper positioned as in Figure 9.2. Dot A is now inside, but you should be able to see it through the paper. Use a pen or pencil to go over the dot heavily. Opening up the paper, you should now be able to see a mark for the true image of A. Label the reflection of A in m, A' . How accurate was your guess about the location of A' ?

Place another dot, B, and guess where its reflection in m ought to be. Now find the image of B under the reflection in m just as you found A' . Call the image of B, B' .

Draw a line segment between A and B, A' and B' . Using an opening of your compass, check to see whether the length of segment $\overline{AB}$ is the same as the length of segment $\overline{A'B'}$. (Henceforth, we will use the symbol "AB" to mean "the length of segment $\overline{AB}$.")

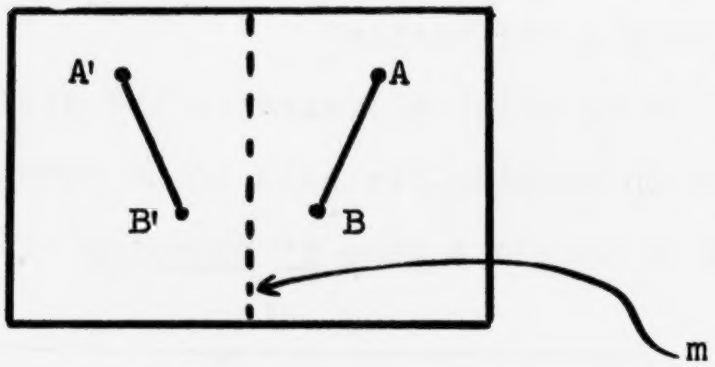


Figure 9.3

Place another point on the same half, call it "C," and try to guess where its reflection in m, C', is. Check by folding on m. Compare the length of $\overline{AC}$ with $\overline{A'C'}$ and of $\overline{BC}$ with $\overline{B'C'}$. Do your measurements support the generalization for Activity 1?

Join A to A' and mark the point where the line drawn crosses m, "A_1" (read: "A one"). How do the lengths of $\overline{AA_1}$ and $\overline{A'A_1}$ compare? Join B to B', C to C' crossing m in B_1 and C_1, respectively. How do $\overline{BB_1}$ and $\overline{B'B_1}$ compare in length? $\overline{CC_1}$ and $\overline{C'C_1}$? What generalization might you make from these observations?

The mapping with respect to a fixed line, m, that takes every point into its mirror image (such as A into A'), is called a reflection in m. You noticed above that the length of $\overline{AB}$ was the same as the length of $\overline{A'B'}$, the length of $\overline{AC}$ was the same as the length of $\overline{A'C'}$, and the length of $\overline{BC}$ was the same as the length of $\overline{B'C'}$. The mapping which assigned A to A', B to B', and C to C' was such that the distance between any two points of its domain was the same as the distance between the images of these points in the range. A mapping like this, which preserves distances, is called an isometry ("iso" means equal, "metry" means measure). Do you think that every reflection is an isometry? Is every isometry a reflection?

Figure 9.4, which illustrates the directions given above for the reflection mapping, is said to be symmetric with respect to line m. m is called a line of symmetry of the figure.

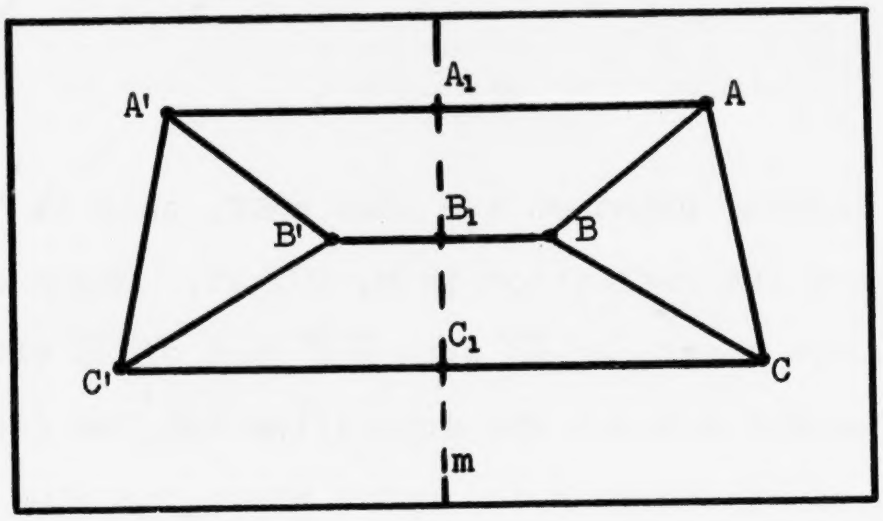


Figure 9.4

What is a line of symmetry for the following kite figure?

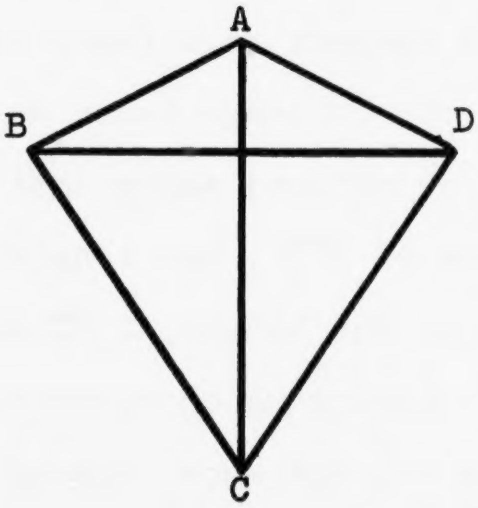


Figure 9.5

How many lines of symmetry does a rectangle have? A square? A circle? (A line of symmetry need not itself be a part of the given figure.)

Returning to the original sheet, illustrated by Figure 9.4, join A_1 to B and B'. Compare A_1B with A_1B'. Join A_1 to C and C'. Compare A_1C and A_1C'. Join any other point, P, on the crease m to A and A', C and C'. What seems to be true about the distances of any point on m to a point and its reflection?

Your observations should lead you to believe that a line reflection is an isometry, and that a figure together with its reflection is symmetric with respect to the line of reflection.

Activity 3: Fold an unlined sheet. Open up the sheet and put a dot on one side of the crease; label it "A."

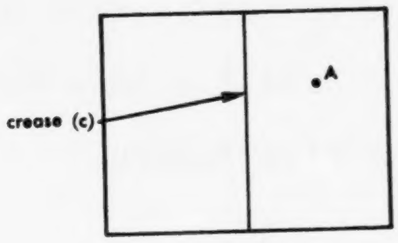


Figure 9.6

Simply by folding this paper, try to locate the reflection of A in m. Some hints are:

(1) Fold back along the crease, and then fold back at A as shown in this figure.

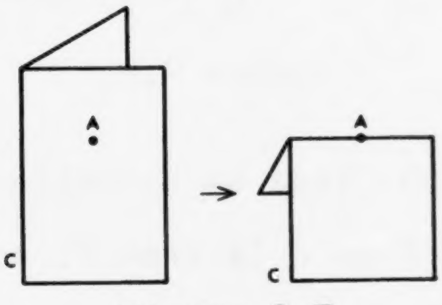


Figure 9.7

Can you finish now?

(2) Fold back once again at A.

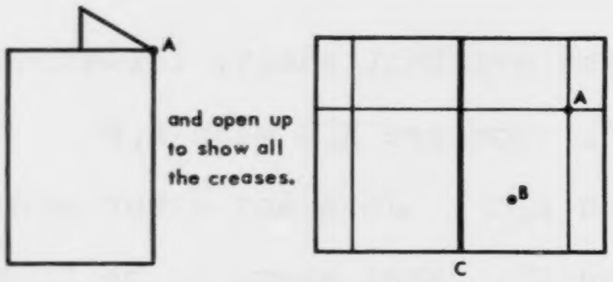


Figure 9.8

Where is A' ? Find B' the same way.

Activity 4: We shall now see how to obtain the reflection of a point in a line without folding. First try to figure out a way yourself. There are many ways of doing it. You will probably need your compass.

One method of finding the reflection of a point A in m is to think of the kite figure. Find 2 points in m, call them P and Q, and think of PAQ as half a kite figure.

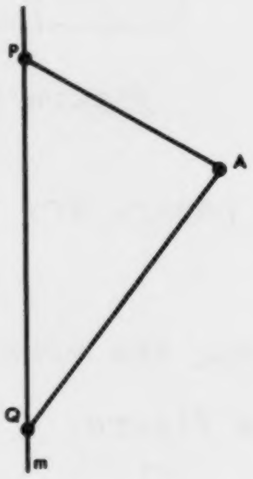
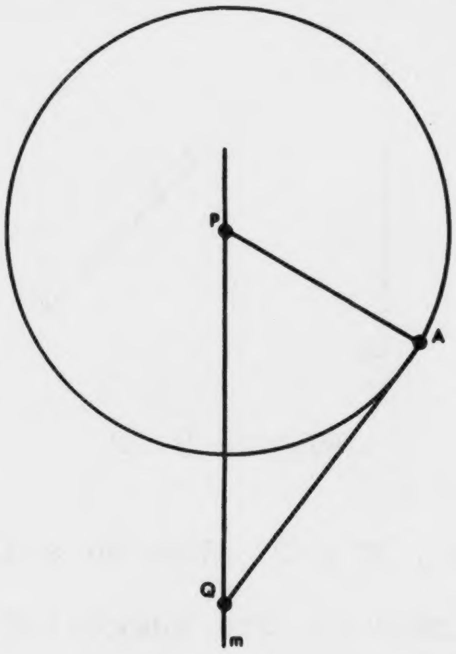


Figure 9.9

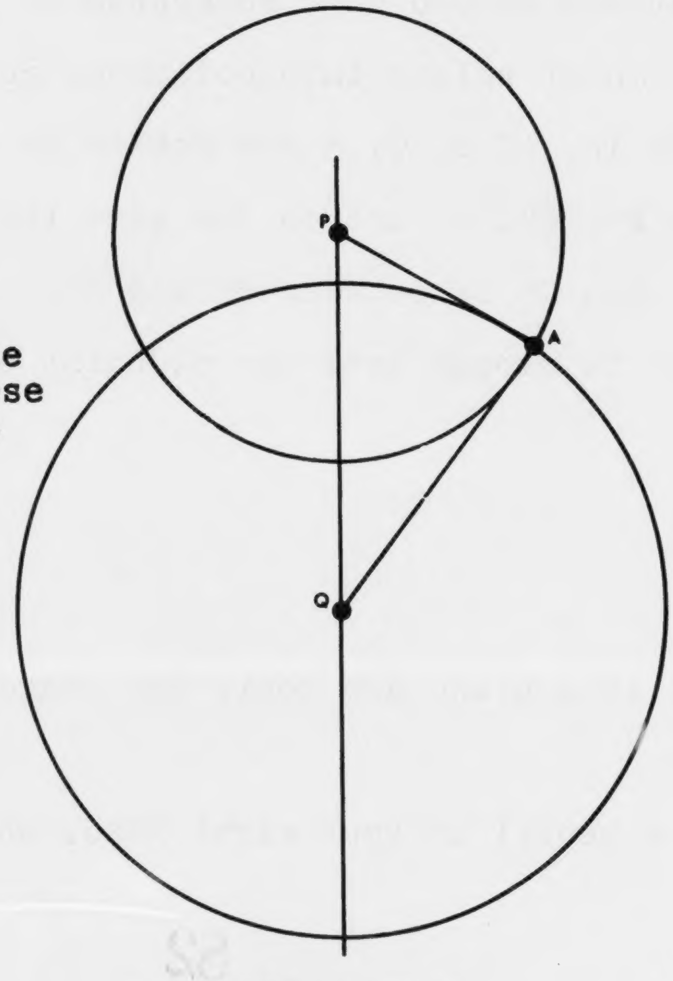
Our previous observations lead us to believe that A' , the image of A, is just as far from P as A is from P, and that A' is just as far from Q as A is from Q. If we draw a circle with P as center and radius of length PA, then A' must be someplace on this circle.



A' is someplace on this circle

Figure 9.10

A' must also be on a circle with center Q and radius QA.



A' is someplace on both of these circles. What point is A'?

Figure 9.11

Join A' to P and Q to complete the kite figure

Using this method of obtaining reflections, find the reflections of points A, B, C if A, B, C are on the same line with B between A and C.

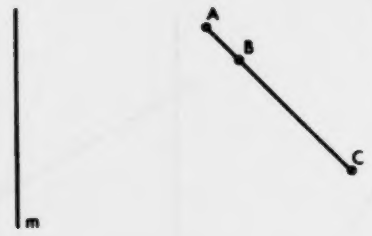


Figure 9.12

Are the image points A', B', C' also on a line? Is B' between A' and C'? What generalizations are suggested by your observations? Suppose D is taken as the midpoint of $\overline{AC}$, what is your guess about D'? Check your guess with a compass.

Your observations should have suggested to you that a reflection maps collinear points into collinear points preserving betweenness. That is, if P, Q, R are points on the same line, ℓ, then their images P', Q', R' are on the same line ℓ'. If Q is between P and R, then Q' is between P' and R'. In fact, the midpoint of a segment is mapped into the midpoint of the image of this segment.

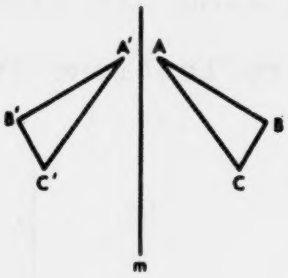
9.3 Exercises

1. Which points in a plane are their own images under a line reflection?
2. If you hold a pencil in your right hand, which hand does it

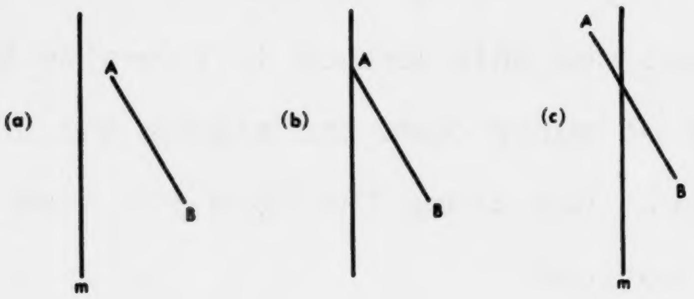
look like in the mirror?

3. If you spin a top clockwise, what does it seem to be doing in the mirror?

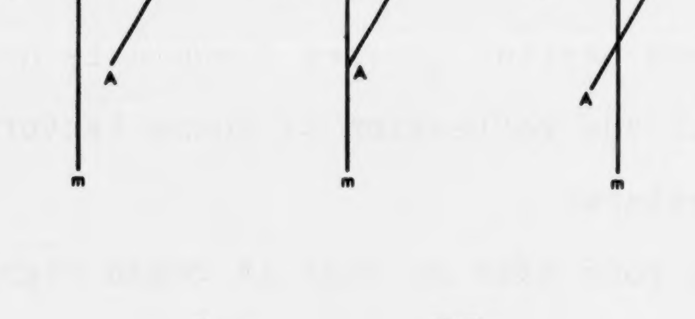
4. If points A', B', C' are the images of points A, B, C under a reflection in m, what are the images of A', B', C' under this reflection?



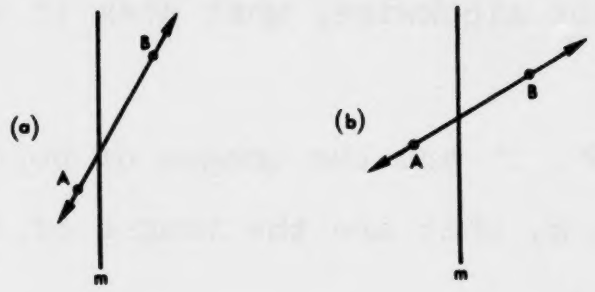
5. Copy the diagrams and draw the reflection in m of the line segment $\overline{AB}$ in each case.



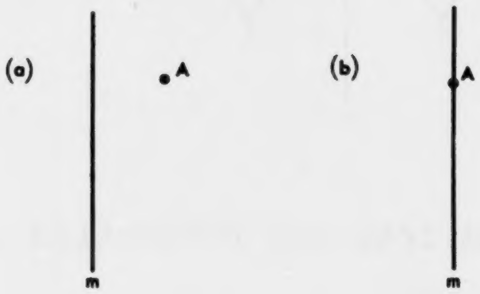
6. Copy the diagrams and draw the reflection in m of ray $\overrightarrow{AB}$ in each case. (Ray $\overrightarrow{AB}$ is the halfline starting at A and passing through B.)



7. Copy the diagrams and draw the reflection in m of line $\overleftrightarrow{AB}$.

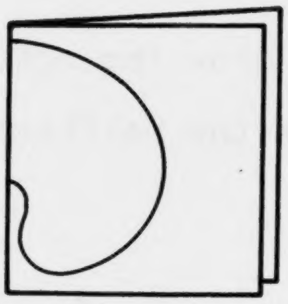


8. Copy the diagrams and find all lines through A that are identical with their reflections in m:



9. Do Exercise 8 by creasing a paper on which m and A are shown, if you did not use this method in Exercise 8.

10. Fold a sheet of paper down the middle and draw some picture as shown here. Cut along the line you drew and open up. What do you notice?

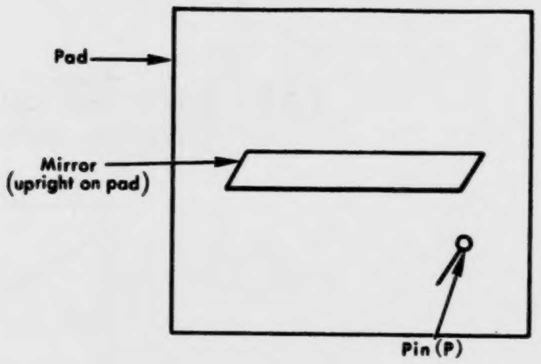


11. Which printed capital letters frequently have a line of symmetry? Will the reflection of these letters in any line be the same letters?

12. Try writing your name so that it reads right in a mirror.

13. Place a sheet of carbon paper under a sheet of paper so that the carbon faces the back side of your paper. Write your name. Look at the back side of your paper in a mirror. What do you see?

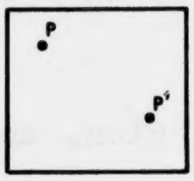
14. For this exercise you will need a pad, 2 pins, and a mirror about $\frac{1}{2}$" wide and at least 6" long.



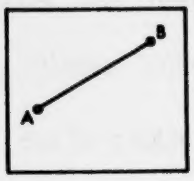
Secure the mirror in an upright position on the pad. (Brace it with a book, or fasten it with pins, scotch tape, or adhesive tape.) Stick a pin upright into the pad about 2" in front of the mirror. Place your eye close to the pad so that you can see the image of the lower part of the pin, P, in the mirror. Try to place the other pin, P' , so that it will always line up with P and the image of P you see in the mirror no matter how you change your line of vision. Where is P' in relation to P? Your pin, P' , should be located at the reflection of P in the mirror. P' is now the image of P under a reflection in the mirror. This close analogy between a reflection mapping and reflections in a real mirror is the reason for using the words "reflection" and "image."

15. By folding your paper, find the line m, for a reflection that will map

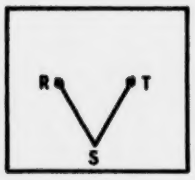
(a) P onto P',



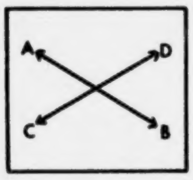
(b) $\overline{AB}$ onto itself,



(c) $\overline{SR}$ onto $\overline{ST}$,



(d) Line $\overleftrightarrow{AB}$ onto line $\overleftrightarrow{CD}$.
(There are 2 lines m)



(e) In each of the above exercises what can you say about the crease?

9.4 Lines, Rays and Segments

Although we picture a line as a taut string, as the edge of a molding, as a mark on the blackboard or paper, we must recognize that these things are quite inaccurate as representations of a line. For example, a string may sag or have a "belly." A string has thickness. A string does not go on and on in both directions endlessly. However, a line has no "belly," no thickness, and does go on endlessly in both directions. But how can we do any better? A line is an idea (like a number), while a physical representation is a thing (like a numeral) used to denote the idea. The marks we call "lines," only represent lines yet we continue to refer to the marks as "lines" because we are not really concerned about the marks but about the ideas the marks represent.

If "A" and "B" name two points of a line then "$\overleftrightarrow{AB}$" names the line containing A and B. We assume that there is only one line (our lines are always straight) that contains two different points. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ are the same line.

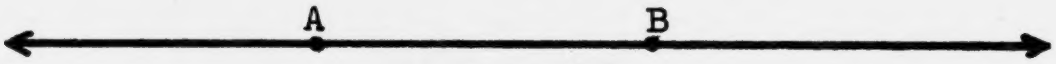


Figure 9.13

We often place arrow heads at the ends of our marks to remind us that the lines are endless in both directions. Sometimes, we place a letter near the mark and refer to the line by the letter.

Consider a line m and a point P in this line:

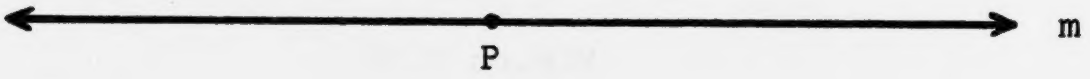


Figure 9.14

The set of points in line m to the right of P, together with P, is a ray. The set of points in m to the left of P together with P is also a ray. Point P is called the endpoint of both rays. Any point, P, in a line together with all the points of the line that are on the same side of P, constitute a ray.

We often name a ray by two capital letters. The left letter names the endpoint of the ray and the right letter names any other point of the ray. An arrow pointing to the right is placed over both letters.

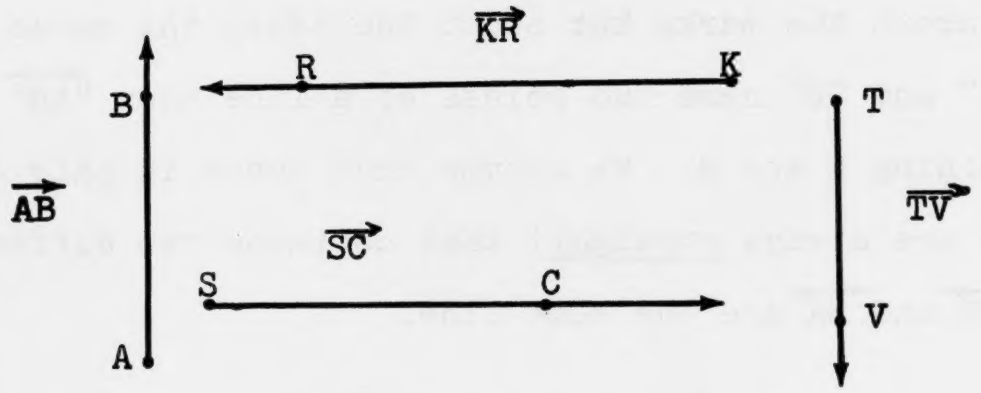

$\overrightarrow{KR}$
R
K
B
T
$\overrightarrow{AB}$
$\overrightarrow{TV}$
$\overrightarrow{SC}$
S
C
V
A

Figure 9.15

If P and Q are two points on line m, $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are different rays. They overlap on a set of points containing P, Q and all the points between P and Q.

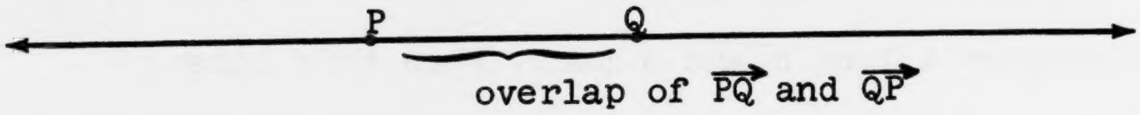

P
Q
overlap of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$

Figure 9.16

The overlap of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ is the segment $\overline{PQ}$ (or $\overline{QP}$).

9.5 Exercises

1. Let A, B, C be any 3 points that are not on the same line (non-collinear points). Draw all the lines you can, each containing two of these points.
   - (a) How many lines did you get?
   - (b) Name the lines.
   - (c) Name each of these lines another way using the same letters.
2. Let A, B, C, D be any 4 points, no three of which are col-

linear. Draw all the lines you can each containing two points.

(a) How many did you get?

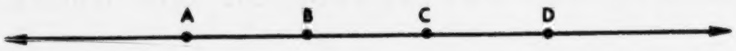
(b) Do the same thing for 5 points, no 3 of which are collinear.

(c) Copy the table below, fill in the blanks, and try to discover a pattern that you feel should continue.

Number of Points	2	3	4	5	6
Number of Lines					

(d) Try to give an argument to support your generalization.

3.



(a) Name the line shown in as many ways as you can using the names of the given points. There are 12 possible ways.

(b) Name all the different rays you can find in the figure. Note $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are all the same ray.

(c) How many different rays did you find?

(d) Copy the table and fill in the blanks.

Number of Points on a Line	1	2	3	4	5
Number of Rays					

(e) Try to discover a pattern that you feel ought to continue.

(f) Try to give an argument to support your generalization.

(g) Name all the segments formed by points A, B, C, D.

(h) How many different segments did you get?

(i) Copy the table and fill in the blanks.

Number of Points on a Line	2	3	4	5	6
Number of Segments					

(j) Try to discover a pattern that you feel ought to continue.

(k) Try to give an argument to support your generalization.

9.6 Perpendicular Lines

In one of the exercises you were asked to find a line, n, through A that is its own reflection in m. Your line should look like the one in Figure 9.17. Whenever we have two lines such that either is its own reflection in the other, we say that these lines are perpendicular to each other. We use the symbol "$\perp$" for "perpendicular" or "is perpendicular to." For Figure 9.17, we have $m \perp n$ and $n \perp m$.

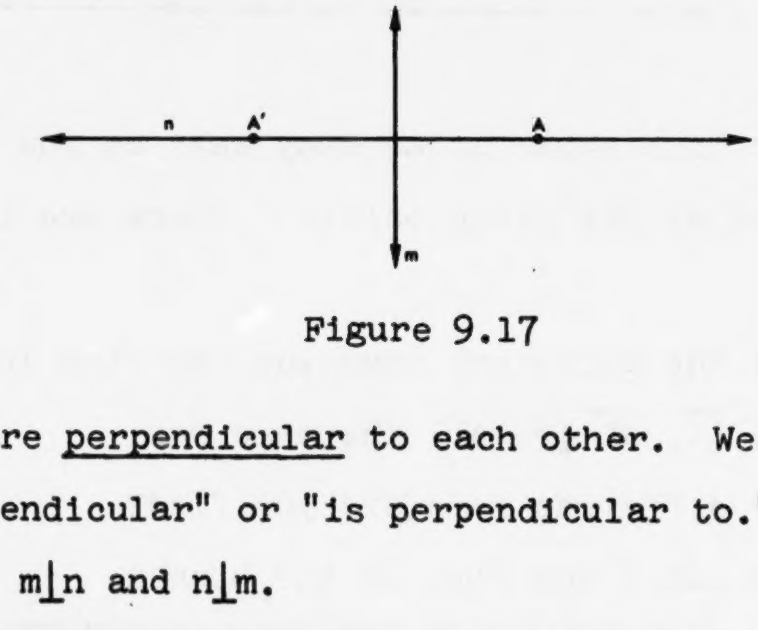


Figure 9.17

If B and B' are two points, each the reflection of the other in line m, then $\overleftrightarrow{BB'} \perp m$, and $m \perp \overleftrightarrow{BB'}$.

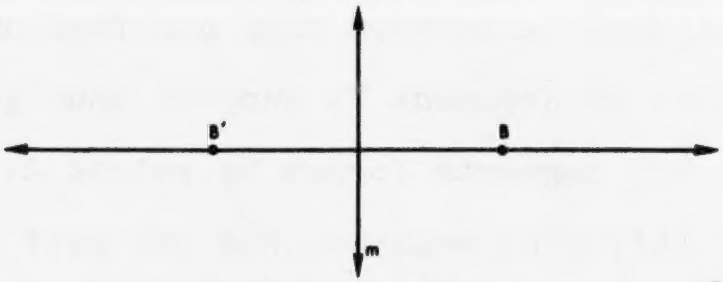


Figure 9.18

We often indicate in a drawing that 2 lines are perpendicular by a little square where the lines cross.

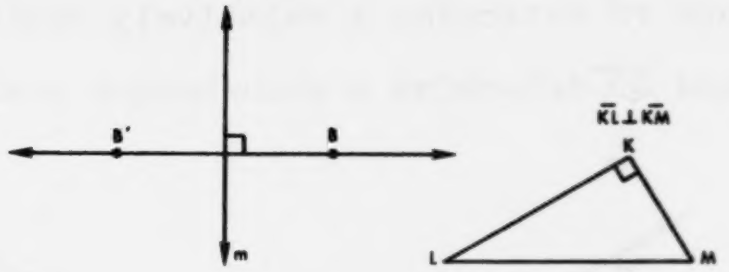


Figure 9.19

Line segments which are in perpendicular lines are said to be perpendicular. Rays which are in perpendicular lines are said to be perpendicular. In fact, any pair such as ray and segment or line and ray are perpendicular if they are in perpendicular lines. We continue to use "$\perp$" for any such perpendicularity.

9.7 Rays Having the Same Endpoint

In this section we shall be dealing with rays that have a common endpoint.

$\overrightarrow{PA}$ and $\overrightarrow{PB}$ are rays with the same endpoint, P.

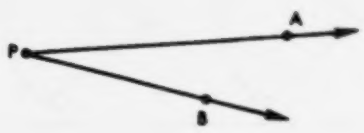


Figure 9.20

If two rays with the same endpoint constitute a line, they are called opposite rays. The rays $\overrightarrow{RC}$ and $\overrightarrow{RD}$ are opposite rays.

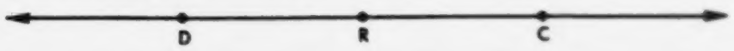


Figure 9.21

An interesting property of a pair of rays with common endpoint is the measure of the angle these rays determine. In Figure 9.20 rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ determine a relatively small angle. In Figure 9.22 rays $\overrightarrow{QC}$ and $\overrightarrow{QD}$ determine a much larger angle.

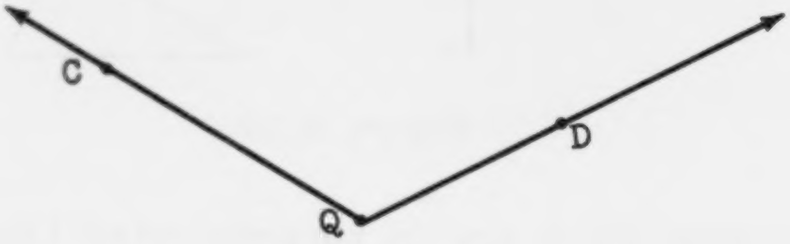


Figure 9.22

If we were given two such pairs of rays with a common endpoint, how could we compare the measures of the angles they determine? To see when such information would be useful, consider the following situation.

Mom makes delicious pies of uniform thickness. She is very skillful at cutting sections from the center. When you get home one day you see these two pieces in a pan.

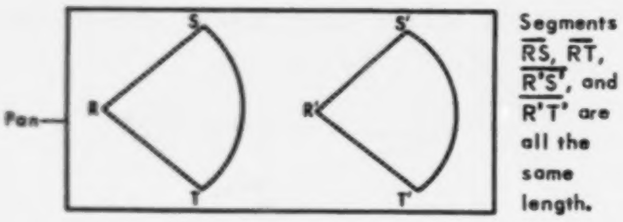

Pan
S
R
T
S'
R'
T'
Segments $\overline{RS}$, $\overline{RT}$, $\overline{R'S'}$, and $\overline{R'T'}$ are all the same length.

Figure 9.23

Which one would you select if you want the larger piece? You may want to use your compass to help you decide. How might you use it? Think about this question a moment before reading on.

If you thought of comparing the distance from S to T with the distance from S' to T', then you have anticipated the text. These measurements were intended to be identical, although the left piece may look larger.

Using this example as a clue, how could you decide which of the following pair of rays determines the greatest angle?

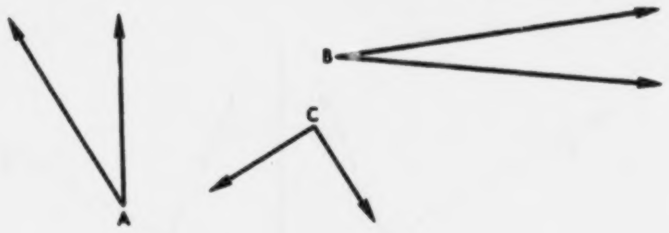


Figure 9.24

Is it the pair of rays at A? at B? at C? Which pair of rays determines the smallest angle?

One way of telling is to draw an arc of a circle across each ray, using in turn points A, B, and C as centers. Each arc should have the same radius (or opening of your compass). After the arcs are drawn, compare the distance between intersection points just as you did for the pie. You will find that $A_1A_2 > B_1B_2$, and $A_1A_2 < C_1C_2$.

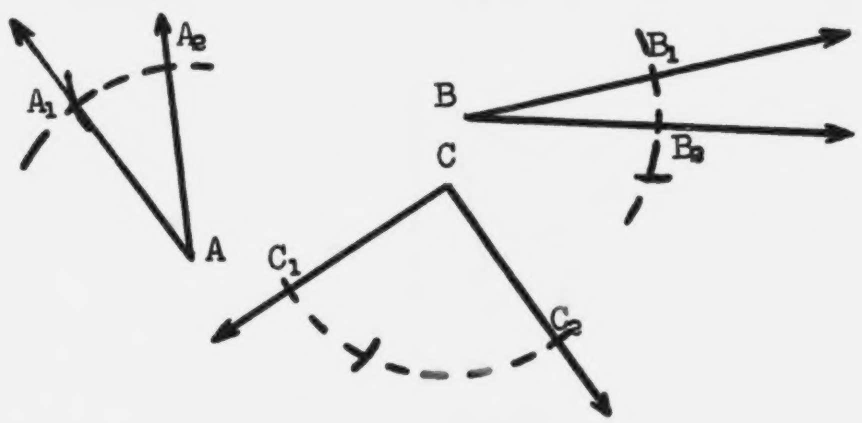


Figure 9.25

This shows that the measure of $\angle C$ is greatest, the measure of $\angle B$ smallest, and the measure of $\angle A$ intermediate.

In Chapter 10 you will measure angles using a protractor, an instrument designed specifically for measuring angles.

Activity 5: On a sheet of unlined paper, draw line m and a pair of rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ as shown:

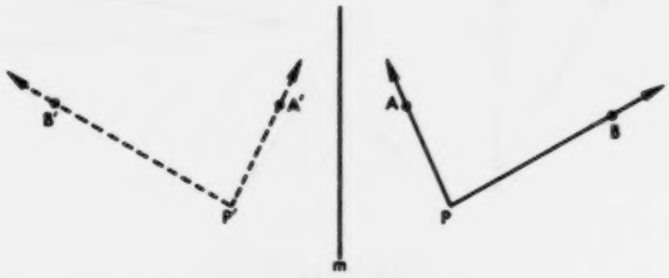


Figure 9.26

Find the reflections $\overrightarrow{P'A'}$ and $\overrightarrow{P'B'}$ of the rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ in m. Guess how the angles formed at P and at P' compare in measure. Check your guess with a compass. Then repeat the experiment with rays meeting in a different angle. What generalization seems to hold?

Activity 6: On a sheet of unlined paper draw line m and join the non-collinear points A, B, C. The figure formed is called "triangle ABC." Find the reflection of triangle ABC in m.

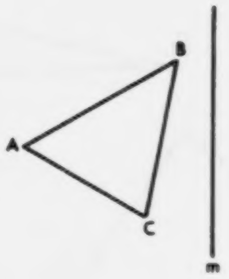


Figure 9.27

Compare the angles at A, B, and C with those at A' , B' , and C' . Then compare the lengths of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ with the lengths of $\overline{A'B'}$, $\overline{A'C'}$, and $\overline{B'C'}$.

Cut out triangle ABC. See if you can make it fit on A' B' C' . Did you have to turn ABC over before making it fit? Will it always be necessary to turn over? If not, when will it be unnecessary? Try this experiment again with a different triangle. Try special kinds of triangles.

Activity 7: Now we are going to make a reflection and then a reflection of the image of this reflection, but in a different line. Draw the following on your unlined paper: triangle ABC and parallel lines m and n.

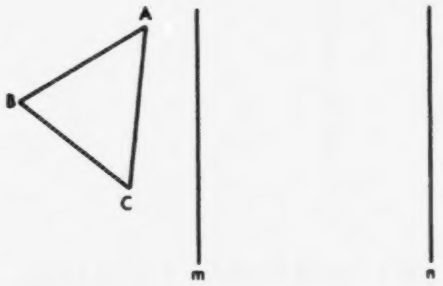
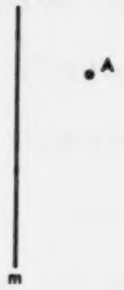


Figure 9.28

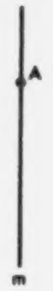
Find the reflection of triangle ABC in m. Call it triangle A' B' C' . Now find the reflection of triangle A' B' C' in n. Call this new figure triangle A"B"C". Try to make some generalizations about the triangles ABC, A' B' C' , and A"B"C". Cut out the three figures. Do they fit? Should they fit? Why do you think so? Which triangles can be made to fit without turning over?

9.8 Exercises

1. (a) Copy the diagram shown and find the line containing point A that is perpendicular to line m. You may try folding your paper.



   (b) Suppose now that A is on m. Copy the diagram below and find the line containing point A that is perpendicular to m. You may want to try folding your paper.



   (c) Try to do (a) and (b) without folding.

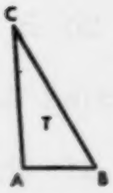
2. (a) What can you say about a triangle that has exactly one line of symmetry?

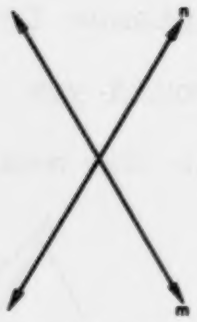
   (b) Can you find a triangle that has just two lines of symmetry?

   (c) Can you find a triangle that has just three lines of symmetry?

   (d) Are there triangles that have more than three lines of symmetry?

3. Copy these figures.



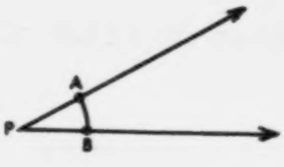


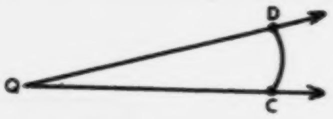
(a) Find the reflection of triangle T in m, call it "T_m", and the reflection of T_m in n, call it "T_{mn}", and finally, the reflection of T_{mn} in m, "T_{mnm}". Compare T, T_m, T_{mn}, T_{mnm}.

(b) Carry out the same steps with m and n perpendicular lines. What can you say now that seems to be true?

4. What is wrong in each of these cases?

(a) The distance from A to B is less than the distance

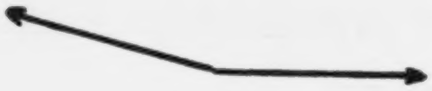


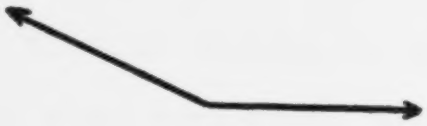


from C to D. Hence, the angle at P is smaller than the angle at Q.

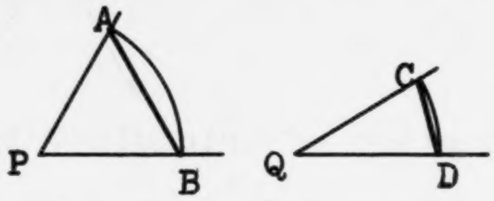
(b) If two triangle cutouts fit then three pairs of angles (one from each triangle) must have the same measure. Hence, if the measures of pairs of angles for two triangles are the same, their cutouts should fit.

5. Why are comparisons difficult for the angles formed by rays that are close to being opposite rays?





6. (a) If the distance from A to B is twice the distance from C to D, would you say that the measure of the angle at P is twice the measure of the angle at Q?



(b) Compare the measure of an angle determined by two opposite rays and the measure of an angle determined by a pair of perpendicular rays. Is the first measure twice as large as the second?

9.9 Reflection in a Point

Does the parallelogram below have a line of symmetry?

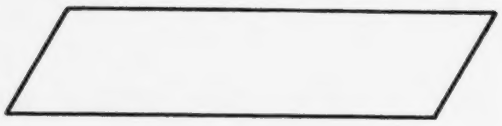


Figure 9.29

In other words, is there a line for which the parallelogram and its mirror image in this line are the same set of points?

After some experimentation, including folding, you will probably say that this parallelogram has no line of symmetry; there is no line reflection that leaves the parallelogram unchanged. However, as we shall soon see, the parallelogram does have a kind of symmetry; it is always symmetric in a point. Try to guess what symmetric in a point means.

Materials needed: Pencil, unlined paper, tracing paper, compass.

Activity 8: Let A, B, and C be points on line ℓ. Let P be any other point (not necessarily on ℓ). Draw ray $\overrightarrow{AP}$ and locate point A' on ray $\overrightarrow{AP}$ so that P is the midpoint of $\overline{AA'}$.

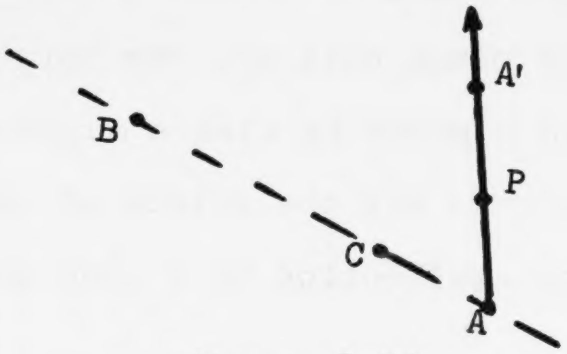


Figure 9.30

We call A' the image of A under the reflection in P. In the same way, find the image of B and C under the reflection in P, calling the images B' and C' respectively. Your figure should resemble Figure 9.31.

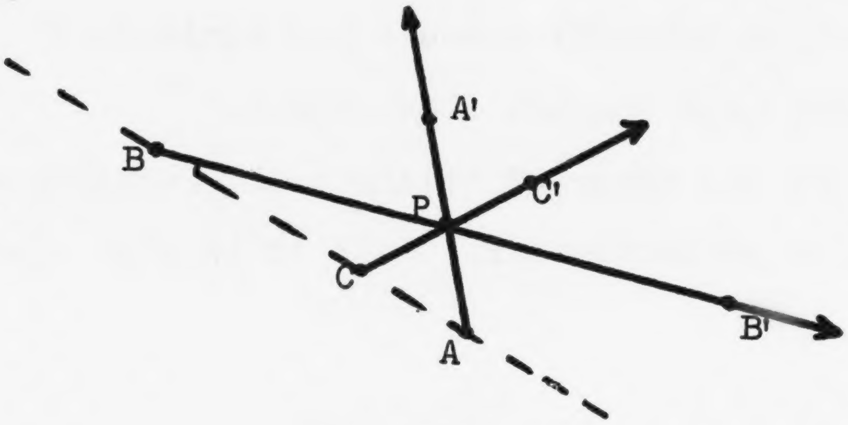


Figure 9.31

Are the points A', B', C' also collinear? Is C' between A' and B'? How does the distance from A to B compare with the distance from A' to B'? Compare the length AC with A'C' and BC with B'C'. What conjectures would you make from this activity regarding: collinearity of points, betweeness, isometry? Try to find a single line in which a reflection maps A into A' and B into B'.

The above activity should have suggested to you the following:

1. Just as a reflection in a line is a mapping of all the points of the plane onto all the points of the plane, reflection in a point is also a mapping of all the points of the plane onto all the points of the plane.
2. Both mappings, reflection in a line and reflection in a point:
   (a) are one-to-one,
   (b) are isometries,
   (c) map collinear points onto collinear points,
   (d) preserve betweeness.

What other properties would you conjecture? Perhaps the next activity will suggest some others.

Activity 9: Find the image of triangle ABC (usually written as "ΔABC") under the reflection in P. Call it ΔA' B' C' where A⟶A', B⟶B', C⟶C'.

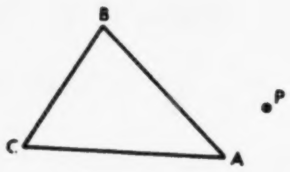


Figure 9.32

Compare the angles at A, B, C with the corresponding angles at

A' , B' , C' . How do the lengths AB, BC, and AC compare with A' B' , B' C' and A' C' ? What additional conjectures would you now make that have not been mentioned regarding the image of a line, ray, and segment under reflection in a point? What conjecture would you make regarding the angle determined by two rays and the angle of their images under reflection in a point?

Have you thought of these:

3. Reflection in a point, just as reflection in a line:
   (a) maps segments onto segments,
   (b) maps rays onto rays,
   (c) maps lines onto lines,
   (d) preserves the measure of the angle formed by two rays.

Cut out ΔABC and ΔA' B' C' . Try to notice exactly what you have to do to make one triangle fit on the other. Do you have to turn one over before they will fit? Recall that for reflection in a line it was often necessary to turn over the figure or its image to obtain a fit.

Materials needed: Pencil, lined paper, unlined paper, compass.

Activity 10: The lines of your lined paper are parallel lines. If two lines are in the same plane and do not cross, the lines are parallel. What happens to parallel lines under a reflection in a line and reflection in a point?

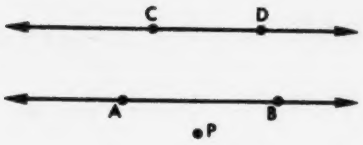


Figure 9.33

Draw two parallel lines and point P as in Figure 9.33. Find the image of $\overleftrightarrow{AB}$ under a reflection in P; call it $\overleftrightarrow{A'B'}$. Does it seem that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel? If they are parallel (let us abbreviate our writing by using the symbol "||" for "is parallel to") we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$. Find the image of $\overleftrightarrow{CD}$ under a reflection in P, calling the image $\overleftrightarrow{C'D'}$. Is $\overleftrightarrow{CD} \parallel \overleftrightarrow{C'D'}$? What conjectures would you be willing to make now?

Next draw three lines: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and m as shown in Figure 9.34.

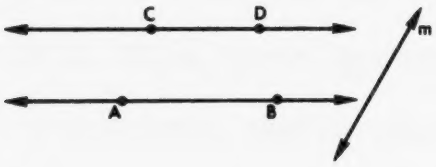


Figure 9.34

Find the reflections of the parallel lines $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ in m. Are the reflections parallel? Is $\overleftrightarrow{CD}$ parallel to its reflection in m? Have you made any of these conjectures?

1. A line maps onto a parallel line under reflection in a point.
2. Two parallel lines map onto two parallel lines under reflection in a point and reflection in a line.

3. The image of a figure under a reflection in a point is a rotation of the figure through a "half turn".

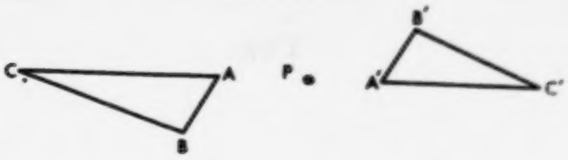
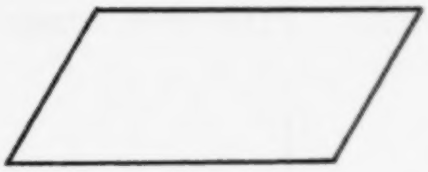


Figure 9.35

9.10 Exercises

1. What point is its own image under a reflection in point P?
2. Is there a point P in which a reflection will map each of the following figures onto themselves? (If there is, show its location.)
   (a) a line segment
   (b) a ray
   (c) a line
   (d) a pair of parallel lines
   (e) a parallelogram

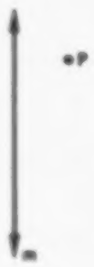


   (f) the letter Z
3. If there is a point in which a reflection will map a figure onto itself, we say the figure is symmetric in a point. If there is a line in which a reflection will map a figure onto itself we say the figure is symmetric in a line. For each printed capital letter in the English

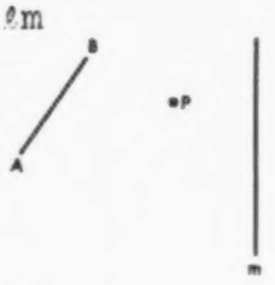
alphabet, decide whether it is symmetric in a point or in a line or neither.

Letter	Symmetric in a Point	Symmetric in a Line	Neither
A	No	Yes	-
B			
C			
.			
.			
.			
.			
Z			

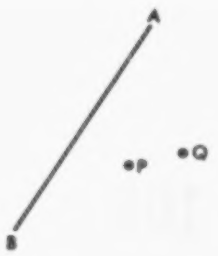
4. Is there a line, m, in which a reflection will map each of the following figures onto itself? (If there is, show it.)
   (a) a line segment
   (b) a ray
   (c) a line
   (d) a parallelogram
5. Using unlined paper and your compass and ruler obtain a line parallel to m. Hint: Find the image of m under the reflection in P.



6. Try to find a way of obtaining a line through P parallel to m (See Exercise 5):
   (a) by folding your paper
   (b) without folding but using your compass and ruler.
7. What kind of symmetry does each of the following have?
   (a) a picture of a face (1) front view (2) side view
   (b) a circle
   (c) a square
   (d) a rectangle
   (e) a picture of a top
   (f) a picture of a five pointed star
   (g) a picture of a six pointed star
   (h) a swastika
   (i) a crescent
8. Denote by "S_P" the reflection mapping in point P, and by "ℓm" the reflection mapping in line m. Find the image of $\overline{AB}$ under each of the following composite mappings:
   (a) ℓm following S_P (c) ℓm following ℓm
   (b) S_P following ℓm (d) S_P following S_P



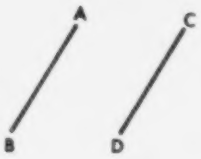
   (e) S_P following S_Q (f) S_Q following S_P



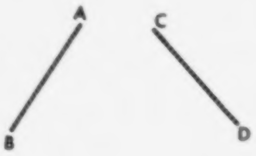
(g) Which of the above mappings (a-f) gave an image of $\overline{AB}$ that was parallel to $\overline{AB}$? (We say that line segments are parallel if they are in parallel lines.)

9. If $\overline{AB}$ and $\overline{CD}$ have the same length, find one or more point refelctions that will map $\overline{AB}$ onto $\overline{CD}$. (You may have to compose two point reflections.)

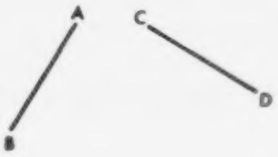
(a)



(b)

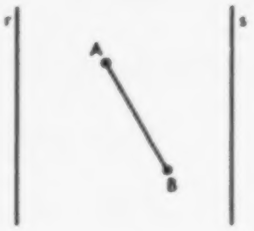


(c)



10. Let r || s. Find the image of $\overline{AB}$ under each of the following composition mappings:

(a) $\ell_r \circ \ell_s$ (b) $\ell_s \circ \ell_r$



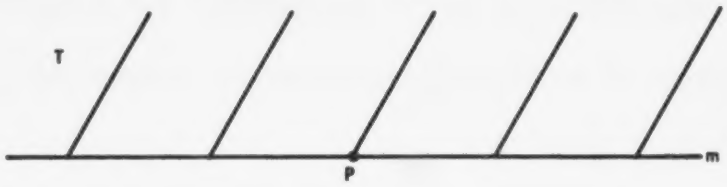
(c) Are the images found in (a) and (b)

(1) the same?

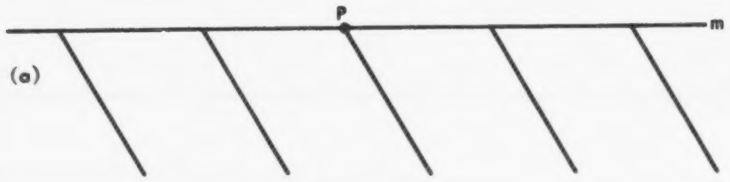
(2) parallel?

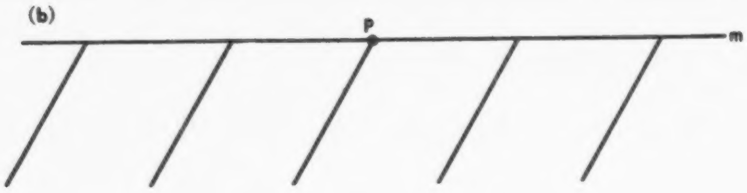
(3) parallel to $\overline{AB}$?

*11. Consider the following design; call it T.



Describe how to obtain each of the following designs, using T or its images under mappings.





12. (a) List at least 5 ways in which a reflection in a line and a reflection in a point are alike.
    (b) List at least 2 ways in which they are not alike.

9.11 Translations

In Chapter 3 one of the basic mappings from W to W was the translation; that is, the mapping with a rule of the form

$$n \longrightarrow n + a.$$

In Chapter 4 these translations were extended to mappings of Z to Z. Then in Chapter 7 a two-dimensional translation was defined from Z x Z to Z x Z.

$$(x,y) \xrightarrow{T_{a,b}} (x + a, y + b)$$

Although $T_{a,b}$ has been defined only for lattice points in the plane, it can be extended naturally to the whole plane.

Activity 11: On a sheet of lined paper select points A, B, C as in Figure 9.36.

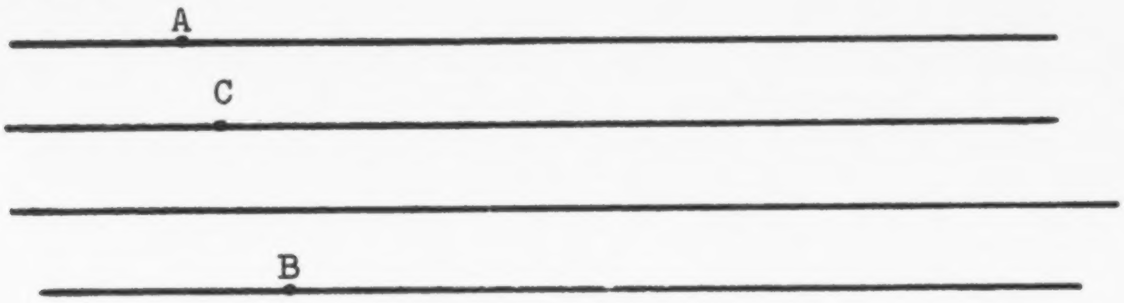


Figure 9.36

Now locate points A', B', C' 3 inches to the right of A, B, C **respectively**. Compare the distances AB with A'B', AC with A'C', BC with B'C'. What can you say about $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$? If C were the midpoint of $\overline{AB}$ what would you conjecture about C'?

Now choose A, B, C as non-collinear points on different lines of the paper as in Figure 9.37. Find the image of ΔABC under the translation "three inches to the right." Call it ΔA'B'C'.

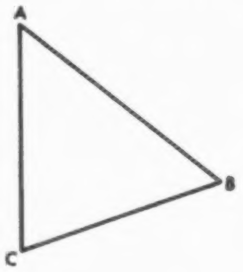


Figure 9.37

Compare the angle at A with that at A' , the angle at B with that at B' , the angle at C with that at C' . What generalizations would you be willing to make for translations regarding: isometry, collinearity, betweeness, midpoints, parallelism, **angles?** Carry out some other activity to check some of your conjectures.

You may have thought of the following generalizations:

A translation

(1) is an isometry;

(2) maps line segments onto parallel line segments;

(3) preserves collinearity, betweeness and midpoints;

(4) preserves parallelism and angle measure.

A translation need not have a magnitude of just three units and a direction only to the right. A translation may have a magnitude of any number of units and any fixed direction. There are infinitely many directions possible for a translation. Because we have the lines of our lined paper so handy, we shall be translating often to the right or left. However, one could always turn the paper so that a translation is along the parallel lines of our paper.

Activity 12: In the lower left hand portion of a piece of tracing paper copy the arrangement of points given in Figure 9.38.

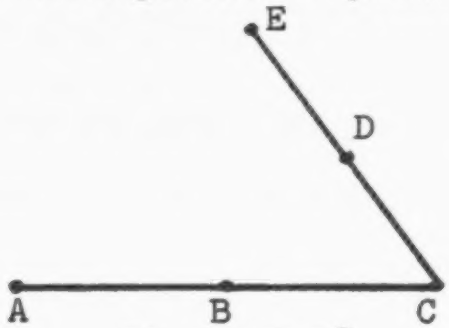


Figure 9.38

Next, using the parallel lines of a sheet of lined paper as a guide, find the images of A, B, C, D, E under the translation "four inches to the right." Then turn the lined paper so that the lines are vertical. Using these lines as a guide, find the images of A', B', C', D', E' under the translation "two inches up." Compare the lengths of $\overline{AB}$ and $\overline{A''B''}$, $\overline{BD}$ and $\overline{B''D''}$, $\overline{AE}$ and $\overline{A''E''}$, $\overline{CD}$ and $\overline{C''D''}$. Compare the lengths of $\overline{A'B'}$ and $\overline{A''B''}$. $\overline{C'D'}$ and $\overline{C''D''}$, and $\overline{B'E'}$ and $\overline{B''E''}$.

Next, draw $\overline{AA''}$, $\overline{BB''}$, $\overline{CC''}$, $\overline{DD''}$, and $\overline{EE''}$. What happens when horizontal and vertical translations are composed? Try to compose two other translations which are not in perpendicular directions.

Your experience in Activities 10 and 11 should have demonstrated the fact that a translation mapping can be defined for the whole plane by simply giving a magnitude and a direction. The identity mapping, that is the transformation that means each point onto itself, is also considered a translation --one with magnitude zero.

9.12 Exercises

1. Which points, if any, are their own images under a translation?
2. Which of the following sets remain the same under some translation? Describe the translation(s).
   (a) a segment

(b) a ray

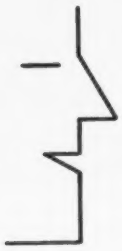
(c) a line

(d) a plane

(e) a half plane

3. Many designs are made by a succession of translations. You can make a face design by doing the following:

(a) Draw a face on a blank sheet, about the size shown here, near the left edge of your paper.



(b) Place a piece of carbon paper face down on another blank sheet.

(c) Mark off 2" intervals along the upper and lower edges of the paper under the carbon.

(d) Line up the paper containing the face figure with the other paper.

(e) Trace over the face figure with pencil.

(f) Move face sheet 2 inches to the right using the marks you made as a guide and trace over face again.

(g) Move face sheet 2 inches again to the right and trace face again.

(h) You should be able to get 4 or 5 faces on your paper this way.

(i) Try to describe the 4 or 5 faces in terms of translations.

4. What happens when you use the same
   (a) line reflection over and over on a figure and its image?
   (b) point reflection over and over on a figure and its image?

9.13 Rotations

We have already observed that a point reflection applied to a figure corresponds to giving the figure a half turn.

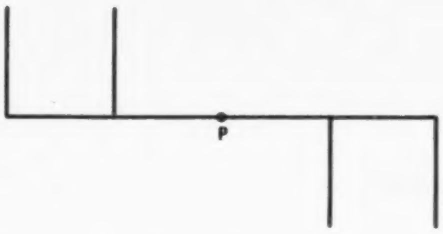


Figure 9.39

If we start with the figure to the left of P and apply the point reflection S_P we obtain the figure to the right of P. If we start with the figure to the right of P and apply S_P we obtain the figure on the left of P. The entire figure above (the original F and its image under S_P) is symmetric in P. But how would you regard the following figure?

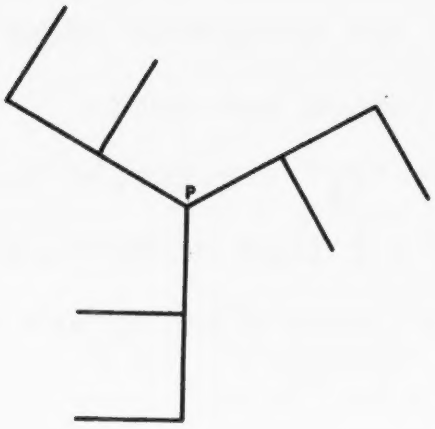


Figure 9.40

Is it symmetric in a line? in a point? It seems to have some kind of symmetry! If we rotate the figure $\frac{1}{3}$ of a complete rotation, we obtain the very same figure. Also, starting with any single F we can obtain the other two by rotating the figure through a $\frac{1}{3}$ turn twice. This suggests mappings which are rotations about some fixed point. A rotation in a point maps every point of the plane onto a point of the plane. What is needed to specify a rotation mapping?

We shall say that a figure has a rotational symmetry if there is a point and a rotation, which is less than a full rotation but not a zero rotation, that maps the figure onto itself. Both "F" figures above have rotational symmetry. Notice that the identity transformation may be regarded as a zero rotation, or as a full rotation.

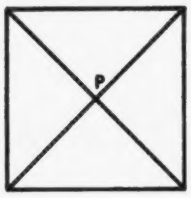
9.14 Exercises

1. Which of the printed capital letters have rotational symmetry?

2. What properties are preserved under a general rotation like a $\frac{1}{3}$ turn? Which are not?

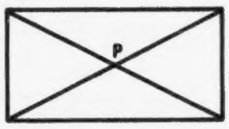
3. Let us denote by "$P_{\frac{1}{4}}$" a rotation that maps every point of the plane by a $\frac{1}{4}$ turn counterclockwise about point P. Which of the following figures are their own images under $P_{\frac{1}{4}}$?

(a)



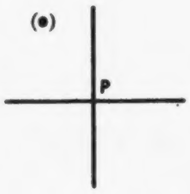
square

(c)

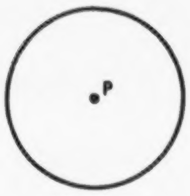


rectangle

(e)

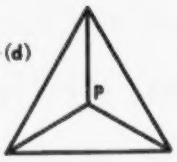


(b)



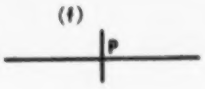
P is center of a circle

(d)



P is center of a triangle with sides of same length

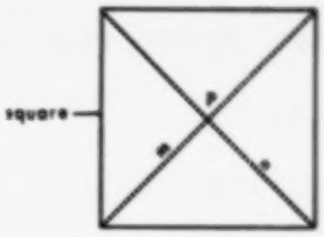
(f)



4. What kind of symmetry or symmetries does each of the following sets of points have?
   (a) lattice points of the first quadrant
   (b) lattice points of the first and second quadrants
   (c) lattice points of the first and third quadrants
   (d) all the lattice points in a plane

5. The various transformations studied in this chapter can be combined to give operational systems where the operation is composition of mappings. Fill in tables (a), (b), (c), (d) showing composition of mappings. In (a), (b), and (c), the sets $\{e, \ell m\}$, $\{e, \ell m, \ell n, S_P\}$, and $\{e, P_{\frac{1}{4}}, P_{\frac{1}{2}}, P_{\frac{3}{4}}\}$ are studied. "e" stands for the identity

mapping and the mappings in (a), (b), and (c) are considered to be acting on the square figure shown here:



(a)

	e	ℓm
e		
ℓm		

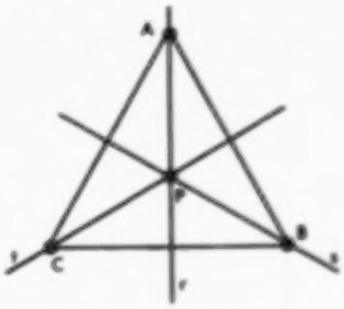
(b)

	e	ℓm	ℓn	S_P
e				
ℓm				ℓn
ℓn				
S_P				

(c)

	e	$P_{\frac{1}{4}}$	$P_{\frac{1}{2}}$	$P_{\frac{3}{4}}$
e				
$P_{\frac{1}{4}}$				
$P_{\frac{1}{2}}$				
$P_{\frac{3}{4}}$		e		

In (d), the mappings are considered to be acting on this equilateral triangle. (All 3 of its sides have the same length.)



r, s, t are fixed lines on the plane.

(d)

	e	$P_{\frac{1}{3}}$	$P_{\frac{2}{3}}$	ℓ_r	ℓ_s	ℓ_t
e						
$P_{\frac{1}{3}}$						
$P_{\frac{2}{3}}$						
ℓ_r						
ℓ_s						
ℓ_t						

6. In 5(a)-(d), find the inverse for each of the mappings:
   (a) 4m (b) S_P (c) $P_{\frac{1}{4}}$ (d) $P_{\frac{1}{3}}$ (e) $P_{\frac{1}{2}}$

7. Which mappings preserve:
   (a) distances
   (b) collinearity
   (c) betweeness
   (d) midpoints
   (e) angle measure
   (f) parallelism

8. Which mappings do not, in general, preserve:
   (a) distances
   (b) collinearity
   (c) betweeness
   (d) midpoints
   (e) angle measure
   (f) parallelism

9. Let us try to extend some of our mappings into 3 dimensions. Describe and try to give examples of the corresponding symmetry for each of the following:
   (a) reflection in a plane
   (b) reflection in a line (in space)
   (c) rotation about a line
   (d) translation in space

10. What are needed to specify each of the following types of mappings:
    (a) a reflection in a line
    (b) a reflection in a point
    (c) a translation
    (d) a rotation

9.15 Summary

1. A reflection in a line is a one-to-one mapping of all the

points of a plane onto all the points of the plane preserving:

distance
collinearity
betweeness
midpoint
angle measure
parallelism

A reflection in a line does not preserve direction. If the reflection of A in m is A', then $\overline{AA'}$ is bisected by m. If m is the line in which a reflection is taken, then each point of m is its own image.

2. A reflection in a point is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance
collinearity
betweeness
angle measure
parallelism
midpoint

A reflection in a point maps a line onto a parallel line; it is the same as a half-turn. If the image of A under a reflection in P is A', then P is the midpoint of $\overline{AA'}$. If P is the point in which a point reflection is taken, then P is the only point that is its own image.

3. A translation is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance
collinearity
betweeness
angle measure
parallelism
midpoint

No point is its own image under a translation which is not the identity mapping.

4. A rotation about a point is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance	angle measure
collinearity	parallelism
betweeness	midpoint

The point about which a rotation is taken is the only point that is its own image, unless the rotation is a multiple of a complete rotation.

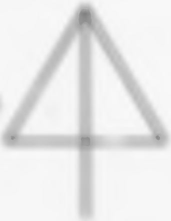
9.16 Review Exercises

1. Fill in the table with "YES," if the mapping always has the property, and "NO," if it does not.

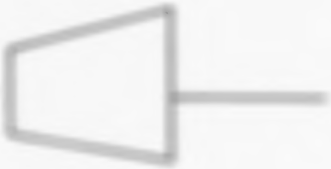
mapping preserves	reflection in a line	symmetry in a point	translation	rotation
distances (isometry)				
collinearity				
betweeness				
midpoint				
angle measure				
parallelism				

2. What kind of mapping and symmetry are suggested by each of the following:

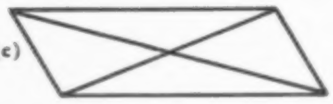
(a)



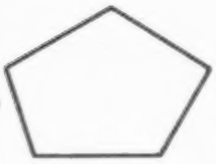
(b)



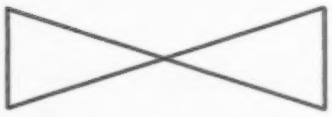
(c)



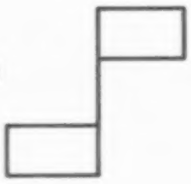
(d)



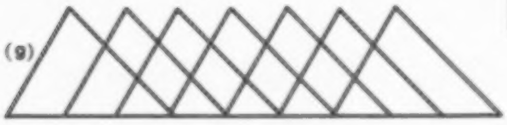
(e)



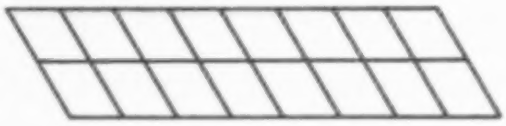
(f)



(g)



(h)



3. Which points are their own images under a
   (a) reflection in a line?
   (b) reflection in a point?
   (c) translation?
   (d) rotation?

4. Which of the following figures may be identical with its image under one of the four mappings mentioned in Exercise 3? Explain.
   (a) a line
   (b) a ray
   (c) a line segment
   (d) two rays which are not opposite yet share a common end point
   (e) a square
   (f) a rectangle
   (g) a parallelogram

5. When are two lines perpendicular?

6. What holds for the two lines m and n if

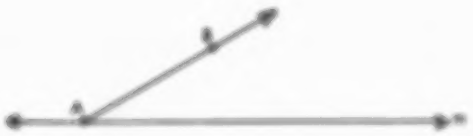
   $S_m \circ S_n = S_n \circ S_m$?

7. Find all points P which have the same image under both composite mappings

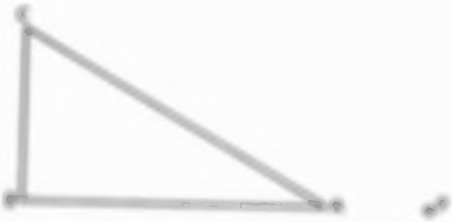
   $\ell m \circ \ell n = \ell n \circ \ell m$.

8. What is the smallest number of line reflections whose compositions suffice to
   (a) map any fixed point A onto a fixed point B?
   (b) map any fixed ray onto any fixed ray?
   (c) map any fixed line onto any fixed line?
   (d) map any fixed line segment onto any fixed line segment of the same length?
   (e) map any ΔABC onto $\Delta A'B'C'$ if $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$?

9. Copy the diagram, and find the reflection of $\overrightarrow{AB}$ in m.



10. Copy the diagram, and find the image of ΔABC under the reflection in point P.



11. Copy the diagram again in Exercise 10, and show the effect of applying P_A, P_B, P_C to ΔABC.

CHAPTER 10

SEGMENTS, ANGLES, AND ISOMETRIES

10.1 Introduction

In previous chapters you have been introduced to many geometrical ideas which have been studied with the help of coordinates and mappings, particularly isometries. In this chapter, we shall tie together many of these results, make them more precise, and extend them to the study of angles.

Since isometries are distance preserving mappings, we shall look more closely at segments and their measure. Then we shall consider angles, how they are measured, and their behavior under an isometry.

We begin by considering some basic properties of lines and planes that are important for our study of segments and angles.

10.2 Lines, Rays, Segments

It may seem to you, on reading this section, that we are making obvious statements and thus wasting time. If so, you will be confusing the obvious with the trivial. Obvious statements can have great significance. For instance, the statement: "The United Stated has only one president" is quite obvious, but its implications for the government and people of the United States are extremely important.

Our first statement about lines is obvious. It is called the Line Separation Principle and it expresses in a precise way the following idea: If we imagine one single point P removed from a line ℓ, the rest of the line "falls apart" into two distinct portions (subsets). Each of these portions is called an open halfline. Along each halfline, one can move smoothly from any point to any other point without ever encountering point P. However, if one moves along line ℓ from a point in one halfline to a point in the other halfline, then it is necessary to cross through point P. See Figure 10.1.

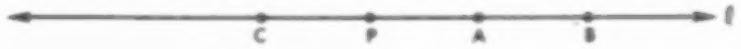


Figure 10.1

The mathematical way of stating this principle more precisely is as follows:

Any point P on a line ℓ separates the rest of ℓ into two disjoint sets having the following properties:

(1) If A and B are two distinct points in one of these sets then all points between A and B are in this set.

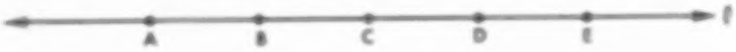
(2) If A is in one set and C is in the other, then P is between A and C.

One of these open halflines may be designated $\overset{\circ\!\!\rightarrow}{PA}$, the other $\overset{\circ\!\!\rightarrow}{PC}$. The little circle at the beginning of the arrow indicates that P itself is not a point of the open halfline. If P is added to $\overset{\circ\!\!\rightarrow}{PA}$ then we obtain the halfline, or ray, designated $\overrightarrow{PA}$ (no circle at the beginning of the arrow). You should be able to name two open halflines of ℓ in Figure 10.1 with point A as the point of separation, and name two distinct rays starting at A. The starting point of a ray is called its endpoint. Note in Figure 10.1 that $\overrightarrow{PA}$ and $\overrightarrow{PB}$ contain precisely the same points, thus $\overrightarrow{PA} = \overrightarrow{PB}$; also ${}^{\circ}\overrightarrow{PA} = {}^{\circ}\overrightarrow{PB}$.

The set of points common to $\overrightarrow{PA}$ and $\overrightarrow{AP}$ is the segment $\overline{PA}$. Thus $\overrightarrow{PA} \cap \overrightarrow{AP} = \overline{AP}$. The set of points found in either $\overrightarrow{PA}$ or $\overrightarrow{PC}$ or both is the line ℓ. Thus $\overrightarrow{PA} \cup \overrightarrow{PC} = \ell$. $\overrightarrow{PA}$ and $\overrightarrow{PC}$ are called opposite rays.

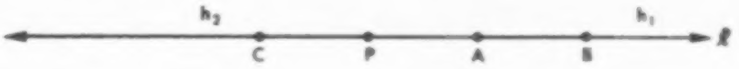
10.3 Exercises

Exercises 1-3 refer to the line ℓ below.



1. Name two distinct rays of ℓ having C as endpoint. Name the open halflines of ℓ for point of separation C.
2. Using two points, if possible, name each of the following:
   (a) $\overline{AB} \cup \overline{BC}$ (e) $\overline{AC} \cap \overline{DB}$ (i) ${}^{\circ}\overrightarrow{BA} \cap \overrightarrow{BC}$
   (b) $\overrightarrow{AB} \cup \overrightarrow{BC}$ (f) $\overrightarrow{AC} \cap \overrightarrow{DB}$ (j) ${}^{\circ}\overrightarrow{BA} \cap {}^{\circ}\overrightarrow{BC}$
   (c) ${}^{\circ}\overrightarrow{AB} \cup \overrightarrow{BC}$ (g) ${}^{\circ}\overrightarrow{AC} \cap {}^{\circ}\overrightarrow{DB}$ (k) $\overline{BA} \cap {}^{\circ}\overrightarrow{BC}$
   (d) $\overrightarrow{AB} \cup {}^{\circ}\overrightarrow{BC}$ (h) $\overrightarrow{AC} \cap \overline{BD}$ (l) $\overline{BA} \cap \overrightarrow{BC}$
3. (a) Name a ray with endpoint B, containing E.
   (b) Name an open halfline contained in $\overrightarrow{BE}$. Are there others?
   (c) Describe the set of points ${}^{\circ}\overrightarrow{CA} \cap {}^{\circ}\overrightarrow{AC}$.
   (d) Name a ray containing ${}^{\circ}\overrightarrow{BD}$. Are there others?
4. Let ℓ be a line and P one of its points. Let h_1 and h_2 be the two open halflines of ℓ determined by P. Let A and B

be distinct points in h_1 and C a point in h_2. Determine whether each of the following statements is true or false



(a) All points of $\overline{AB}$ are in h_1.

(b) All points of $\overleftrightarrow{AB}$ are in h_1.

(c) Either $\overrightarrow{AB}$ or $\overrightarrow{BA}$ contains C.

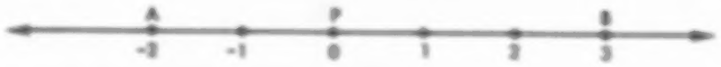
(d) Both $\overrightarrow{AB}$ and $\overrightarrow{BA}$ contain C.

(e) $\overrightarrow{PC}$ contains A.

(f) $\overrightarrow{CP}$ contains A.

(g) All points of $\overrightarrow{PB}$, other than P, are in h_1.

5.



Using the data shown in the above diagram tell what values x may have if x is the number assigned to a point in each of the following sets:

(a) $\overline{AB}$ (c) $\overrightarrow{BA}$ (e) $\overleftrightarrow{AB}$ (g) $\overrightarrow{AP} \cap \overrightarrow{PB}$

(b) $\overrightarrow{AB}$ (d) $\overleftrightarrow{AB}$ (f) $\overline{AB} \cap \overline{PB}$ (h) $\overrightarrow{AP} \cup \overrightarrow{PA}$

10.4 Planes and Halfplanes

A second separation principle concerns planes, and is another example of an obvious statement. It states an essential property of planes.

It will help you to think about a plane if you imagine a very large flat sheet of paper, so large that its edges are inconceivably far and unreachable. In fact, it would be even better if you could think of a plane as having no edges, just as a line has no endpoints. In such a plane we could think of a line; otherwise a line, reaching any edge the paper might have, would have to stop and thus acquire an endpoint. But then it would not be a line!

We cannot draw a line, since any drawing would necessarily have to begin and end. In the same vein we cannot draw a plane. But we suggested a line by drawing a segment and arrows at each end. We suggest a plane by drawing a piece of it, as shown in figure 10.2. Unfortunately there is no easy way to suggest in the drawing that the plane has no edges.

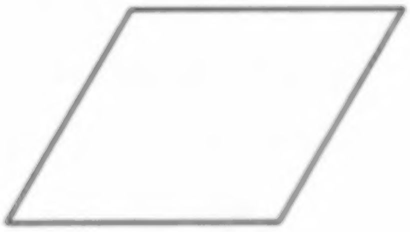


Figure 10.2

However, to remind you that we are talking about a plane, rather than a piece of it, we shall use Greek letters to name the plane. For instance α and β (alpha and beta) will be the names of planes.

Our second separation principle concerns planes. This Plane Separation Principle expresses in a precise manner, the following idea:

Any line ℓ in a plane α separates the rest of the plane into two distinct portions (subsets). Each of these portions is called an open halfplane. Within each halfplane one can move smoothly from any point to any other point without ever encountering line ℓ. However, if one moves within plane α from a point

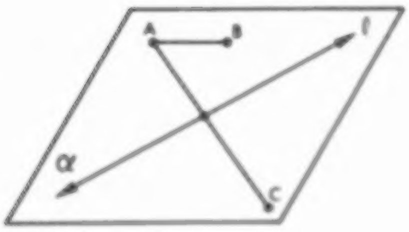


Figure 10.3

in one open halfplane to a point in the other open halfplane, then it is necessary to cross line ℓ. The mathematical way of stating this is as follows:

Any line ℓ in a plane α separates the rest of α into two disjoint sets having the following properties:

(1) If A and B are two distinct points in one of these sets then all points of $\overline{AB}$ are in this set.

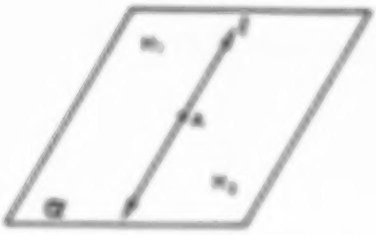
(2) If A is in one set and C is in the other then $\overline{AC}$ (the segment, not $\overleftrightarrow{AC}$) intersects ℓ in a point.

The line ℓ is called the boundary of each open halfplane determined by ℓ, but actually it does not belong to either open halfplane. The union of an open halfplane with its boundary is called a halfplane.

Figure 10.4

In the plane named β in Figure 10.4 you see line m separating β into two halfplanes named H_1 and H_2. If A is in H_1, call H_1 the A-side of m. Then H_2 is the side opposite the A-side.

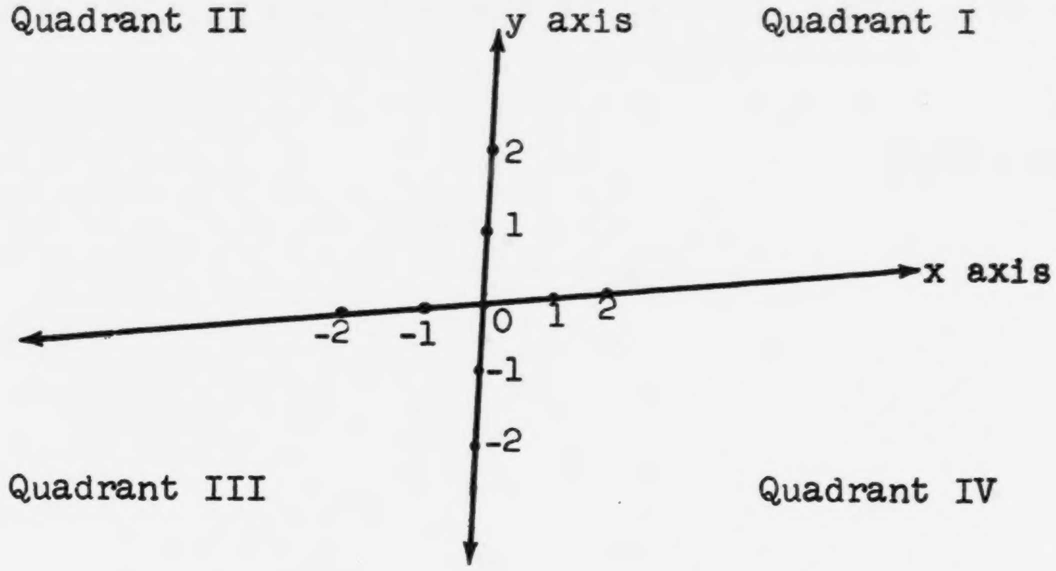
10.5 Exercises



Let σ be a plane containing line ℓ, and let ℓ contain point A. Let the two open halfplanes determined by ℓ be H_1 and H_2. Determine whether each of the following statements is true or false:

1. Any line containing A, other than ℓ, contains points of H_1 and H_2.
2. Any ray with endpoint A, not lying in ℓ, contains points of H_1 and H_2.
3. Any segment containing A as an interior point and not lying in ℓ, contains points of H_1 and H_2.
4. If B and C are any two distinct points in H_1, then $\overline{BC}$ intersects ℓ.
5. If B and C are any two distinct point in H_1, then $\overrightarrow{BC}$ does not intersect ℓ.
6. If B and C are two distinct points in H_2, then $\overleftrightarrow{BC}$ may not intersect ℓ.

7. If D is in H_1 and E is in H_2, then it is possible that $\overleftrightarrow{DE} \parallel \ell$.



8. The coordinate system shown separates the plane into four sets, each called a quadrant. The x-axis separates the rest of the plane into two open halfplanes, one containing the point with coordinates (0,2) the other containing (0,-2). Let us name the first of these open halfplanes H_{+x}, the other H_{-x}. Similarly, the y-axis separates the plane into two open halfplanes which we name H_{+y} and H_{-y}, with the obvious meaning attached to each. Now Quadrant I = $H_{+x} \cap H_{+y}$. In the same manner define Quadrants II, III, IV.

10.6 Measurements of Segments

Let us examine what is involved when we use a ruler to find the length of a segment. We first place the graduated edge of the ruler against a line segment, say $\overline{AB}$, matching the zero point of the ruler with one of the points, say A. (See Figure 10.5.)

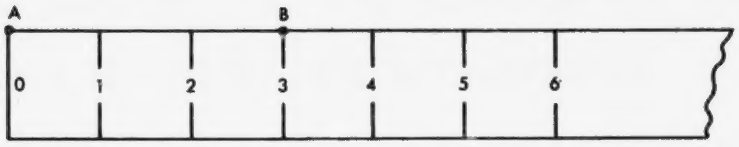


Figure 10.5

We then assign to point B the number on the ruler which matches it and say that the length of $\overline{AB}$, denoted by AB, is the number assigned to B. In our example the ruler assigns 0 to A and 3 to B. So AB = 3.

Now suppose we move the ruler to the left until it arrives at the position shown in Figure 10.6.

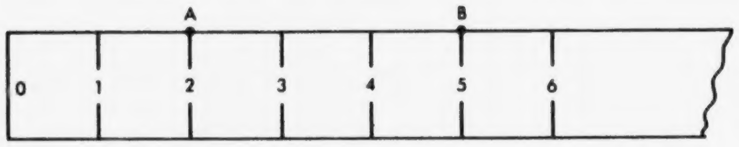


Figure 10.6

What is the number assigned by the ruler to A? to B? Using these numbers how can you find AB? Probably you subtracted 2 from 5 since this calculation gives the number of unit spaces in $\overline{AB}$. But suppose we turned the ruler around to this position.

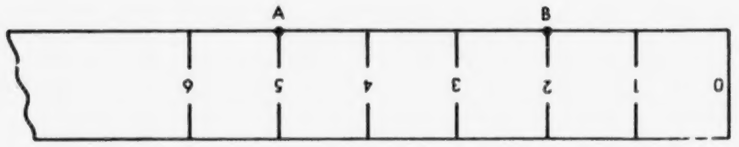


Figure 10.7

What are the assignments made by the ruler to A and B in this position? Would you subtract 5 from 2 to find AB? This, of course gives -3. In measuring the length of a segment we want to know how many unit spaces it contains. Therefore, we use only positive numbers for lengths of segments. If we do subtract 5 from 2, we must take the absolute value of the difference.

In general, then, if a ruler assigns the numbers x_1 and x_2 to the endpoints of a segment $\overline{AB}$, we can use the distance formula.

$$AB = |x_1 - x_2|$$

Let us now consider a ruler which has negative numbers on it (like a thermometer) that is placed against $\overline{AB}$ and looks like this.

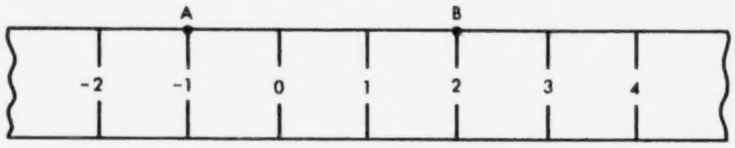


Figure 10.8

or perhaps like this,

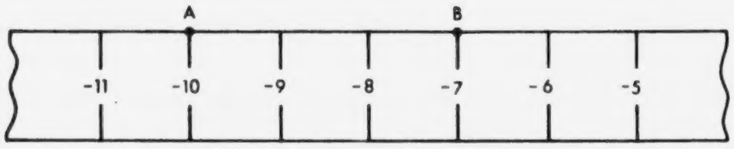


Figure 10.9

or even like this.

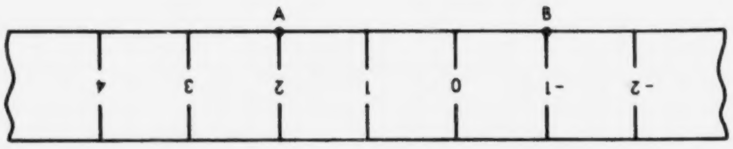


Figure 10.10

Does the distance formula give us the number of unit spaces in each case? Let us see.

For the ruler in Figure 10.8 the formula yields: $AB = |-1-2|$

For the ruler in Figure 10.9 the formula yields: $AB = |-10-(-7)|$

For the ruler in Figure 10.10 the formula yields: $AB = |2-(-1)|$

Is 3 the value of AB in each case?

You know that the distance from A to B should be the same as the distance from B to A. In the formula this reverses $\underline{x_1}$ and $\underline{x_2}$. Is it true that $|x_1 - x_2| = |x_2 - x_1|$?

Let us review the results of this section in terms of mappings.

(a) A ruler assigns numbers $\underline{x_1}$ and $\underline{x_2}$ to the endpoints of $\overline{AB}$. Thus $A \longrightarrow x_1$ and $B \longrightarrow x_2$. Then we say $AB = |x_1 - x_2|$.

(b) Moving the ruler 2 spaces to the left (as we did) is a translation with rule $n \longrightarrow n + 2$. Thus $x_1 \longrightarrow x_1 + 2$ and $x_2 \longrightarrow x_2 + 2$. We ask you to answer two questions:

(1) Does a translation preserve distance?

(2) Is $|x_1 - x_2|$ preserved under this translation?

Suppose the ruler were moved to the right. Are the last two answers changed?

(c) In Figures 10.8 and 10.9 we moved the ruler still further to the left. Is the composition of two translations still a translation? Do the answers to our two questions change?

(d) Let us compare the rulers in Figures 10.6 and 10.10.

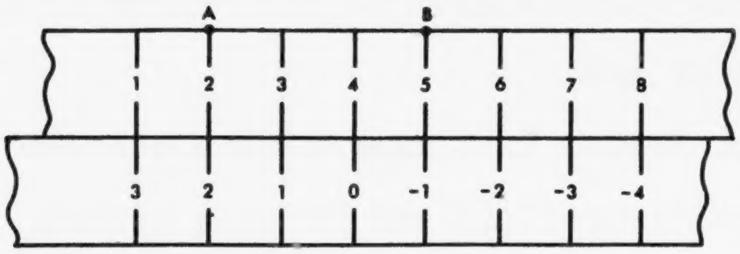


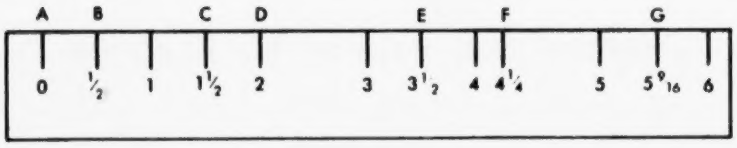
Figure 10.11

Do you see a mapping of Z into Z with the rule $n \longrightarrow 4 - n$? Then $x_1 \longrightarrow 4 - x_1$ and $x_2 \longrightarrow 4 - x_2$. But $|(4 - x_1) - (4 - x_2)| = |x_2 - x_1|$. And again we can say "yes" to our two questions above. We conclude that the distance formula gives the correct distance for all positions of a ruler.

We use the term line coordinate system to describe the relationship between the points and numbers on a number line. The number assigned to the point is called the coordinate of the point in the system. Using these terms we can say that the distance between two points in a line coordinate system is the absolute value of the difference of their coordinates.

10.7 Exercises

1. In this exercise use the numbers assigned by the ruler to points in the diagram below. First express the length of the segments listed below in the form $|x_1 - x_2|$. Then compute the length.



(a) $\overline{AC}$

(b) $\overline{AE}$

(c) $\overline{AG}$

(d) $\overline{FA}$

(e) $\overline{BC}$

(f) $\overline{BD}$

(g) $\overline{FB}$

(h) $\overline{GB}$

(i) $\overline{CD}$

(j) $\overline{FC}$

(k) $\overline{EF}$

(l) $\overline{GF}$

2. A ruler, graduated with negative and positive numbers assigns 0 to point A. What number does it assign to B if AB = 3? (Two answers.)

3. A ruler assigns 8 to D. What number does it assign to E if DE = 2. (Try to solve this problem by solving the equation $|x - 8| = 2$.)

4. A ruler assigns 83 to F. What number does it assign to G if FG = $6\frac{1}{2}$?

10.8 Midpoints and other Points of Division

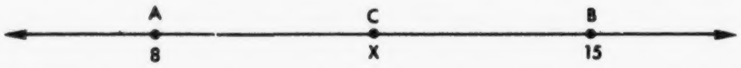


Figure 10.12

Let a ruler assign 8 to A and 15 to B. We shall try to find the number assigned to C, the midpoint of $\overline{AB}$. Let that number be represented by x (See Figure 10.12). You recall that a midpoint of a segment bisects it. This means that the length of $\overline{AC}$ is the same as the length of $\overline{CB}$. This explains statement (1) below. Explain (2). Now x-8 must be positive. Why? Also 15-x is positive. Why? So the equality in (2) implies (3). Explain (4) and (5). Check whether for $x = 11\frac{1}{2}$, AC = CB.

(1) $AC = CB$

(2) $|x - 8| = |15 - x|$

(3) $x - 8 = 15 - x$

(4) $2x = 23$

(5) $x = 11\frac{1}{2}$

Use this method of finding the number assigned to the midpoint of $\overline{DE}$ if in a certain line coordinate system the coordinate of D is -2 and the coordinate of E is 5.

Let us generalize this method; that is, let us find a formula for midpoints. In a line coordinate system let A have coordinate $\underline{x_1}$ and let B have coordinate $\underline{x_2}$, where $x_1 < x_2$, and let C, the midpoint of $\overline{AB}$, have coordinate $\underline{x}$ (see Figure 10.13).

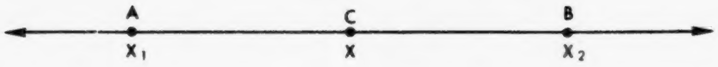


Figure 10.13

Then,

$$AC = CB$$
$$|x - x_1| = |x_2 - x|$$
$$x - x_1 = x_2 - x \quad \text{(Why?)}$$
$$2x = x_1 + x_2$$
$$x = \frac{1}{2}(x_1 + x_2)$$

Do you recognize that $\underline{x}$ is the mean of $\underline{x_1}$ and $\underline{x_2}$? This is an easy way to remember the formula.

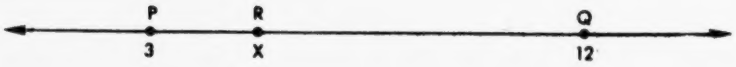


Figure 10.14

Suppose R is in $\overline{PQ}$ and it divides $\overline{PQ}$ in the ratio 1:2 from P to Q. (The phrase "from P to Q" tells that PR corresponds to

1 and RQ to 2.) To find x for the data shown in Figure 10.14 we can proceed as follows:

(1) $\frac{|x - 3|}{|12 - x|} = \frac{1}{2}$ or $2|x - 3| = |12 - x|$

Both $x - 3$ and $12 - x$ are positive.

(2) $2 \cdot (x - 3) = 12 - x$

(3) $2x - 6 = 12 - x$

(4) $3x = 18$

(5) $x = 6$

Check $\frac{|6 - 3|}{|12 - 6|} = \frac{1}{2}$

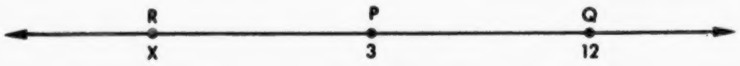


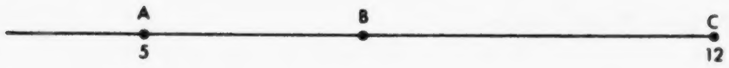
Figure 10.15

Suppose, instead, that R were not between P and Q, but that P is between R and Q, as in Figure 10.15. Then $3 - x$ is positive, and $12 - x$ is positive. Then step (2) above becomes (2') $2 \cdot (3 - x) = (12 - x)$. Complete the solution and check.

10.9 Exercises

In Exercises 1 - 4 you are asked to derive results which are going to be used in later developments. In this respect they differ from other exercises whose results can be forgotten without harm to an understanding of future developments. These exercises are marked "#." In the following sections such exercises will also be marked with the symbol "#."

#1. Let B be an interior point of $\overline{AC}$ and let a ruler assign numbers 5 and 12 to A and C, as shown.



(a) What is one possible assignment to B that *guarantees* that B is an interior point of $\overline{AC}$? Name three other possible assignments to B that also guarantee that B is between A and C. What are all the possible assignments to B such that B is between A and C?

(b) Show that AB + BC = AC if B is assigned the number 8 or the number $11\frac{1}{2}$.

(c) Show that AB + BC = AC if B is assigned the number $\underline{x}$ such that $5 < x < 12$.

This last result may be stated in general as follows: If B is between A and C, then AB + BC = AC. It is called the *Additive Property of Betweeness for Points*.

#2. Suppose two circles in a plane have centers at A and B, and respectively radii $\underline{r_1}$ and $\underline{r_2}$. We are going to compare AB with $r_1 + r_2$ for different positions of the two circles.

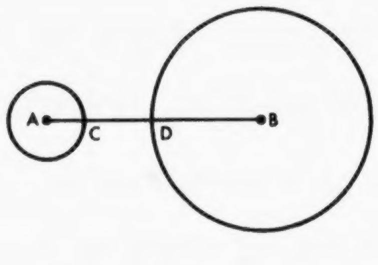


Figure 10.16

(a) Suppose the circles do not intersect as shown in Figure 10.16. Then AB = AD + DB (Why?) and AD = AC + CD. (Why?) So AB = AC + CD + DB. But AC = r_1 and DB = r_2. Hence AB = r_1 + CD + r_2. Thus AB > $r_1 + r_2$.

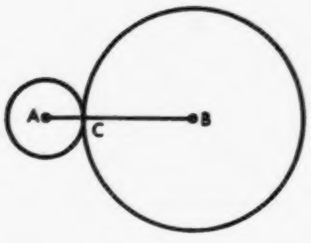


Figure 10.17

(b) Consider the position of the circles in Figure 10.17, in which the circles just touch at C. Show that AB = $r_1 + r_2$.

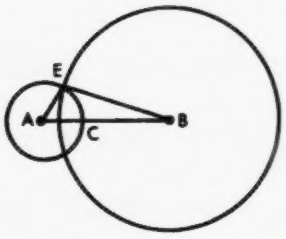


Figure 10.18

(c) Consider the position of the circles in Figure 10.18 in which they intersect. One of the points of intersection is named E. Now AB = AC + CB, (Why?) and CB < r_2 so AB < $r_1 + r_2$. (Why?) $\overline{EA}$ and $\overline{EB}$ are also radii and therefore EA = r_1 and EB = r_2. Therefore AB < EA + EB.

In words, this last result suggests that the length of any side of a triangle (ΔABE in this case), is less than the sum of the lengths of the other two. We call this conclusion the Triangle Inequality Property. You should note that for any triangle, there are three inequalities. Thus, for ΔDEF (Figure 10.19)

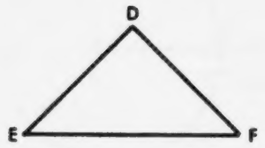


Figure 10.19

(a) DE < EF + FD,

(b) EF < FD + DE,

(c) FD < DE + EF.

#3. For Figure 10.20, we see by the Triangle Inequality Property that in ΔABD, DA + AB > DB. Use this fact to show that the perimeter of ΔDAC is greater than the perimeter of ΔDBC.

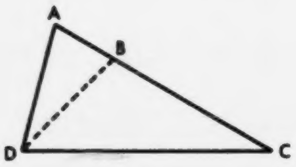


Figure 10.20

#4. Show in any triangle that the difference between the lengths of any two sides is less than the length of the third side.

5. Which of the following triplets of numbers may be the lengths of the sides of a triangle?

(a) 5, 6, 8	(d) 4.1, 8.2, 12.3
(b) 5, 6, 11	(e) 18, 22, 39
(c) 1, 2, 3	(f) $4\frac{1}{2}$, $4\frac{3}{4}$, $4\frac{5}{8}$

10.10 Using Coordinates to Extend Isometries

Let us consider an isometry, f, of a pair of points {A, B}.

If $A \xrightarrow{f} A'$ and $B \xrightarrow{f} B'$, then $AB = A'B'$. How can we extend this isometry to a third point of $\overleftrightarrow{AB}$? This is easily done by working with the line coordinate system on $\overleftrightarrow{AB}$ that assigns 0 to A and 1 to B. Since $AB = A'B' = 1$, there is a coordinate system on $\overleftrightarrow{A'B'}$ that assigns 0 to A' and 1 to B'.

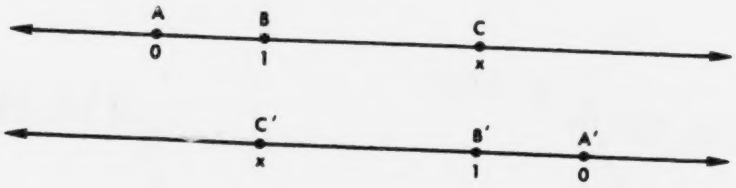


Figure 10.21

Now suppose C is any point on $\overleftrightarrow{AB}$ and let its coordinate be x. We can extend f to C by taking for its image the point C' on $\overleftrightarrow{A'B'}$ whose coordinate is also x. To convince yourself that we have succeeded in extending f you should verify that $AC = A'C'$ and $BC = B'C'$. You can do this by using the distance formula. How can you extend f to other points of $\overleftrightarrow{AB}$?

Before we examine an isometry involving non-collinear points, we will need a plane coordinate system.

Given three non-collinear points A, B, C we can introduce a plane coordinate system (see Figure 10.22) much as a plane lattice coordinate system was introduced in Chapter 7. Take A as origin, $\overleftrightarrow{AB}$ as x-axis, $\overleftrightarrow{AC}$ as y-axis. Assign to B the coordinate 1 on the line coordinate system on $\overleftrightarrow{AB}$, assign to C the coordinate 1 on the line coordinate system on $\overleftrightarrow{AC}$. The coordinate 0 in both systems is assigned to A. Here we equip the axes with line coordinate systems like those we have been using in this chapter (not just lattice points). The coordinates of a point D in the plane are found, as before, by drawing lines through D parallel to the y- and x-axes. The line coordinate of the point E (where the parallel through D intersects the x-axes) becomes the x-coordinate of D, and the line coordinate of F where the other parallel through D cuts the y-axis) becomes the y-coordinate of D. (In Figure 10.22 the coordinates of D are $(1\frac{1}{4}, 1\frac{1}{2})$.)

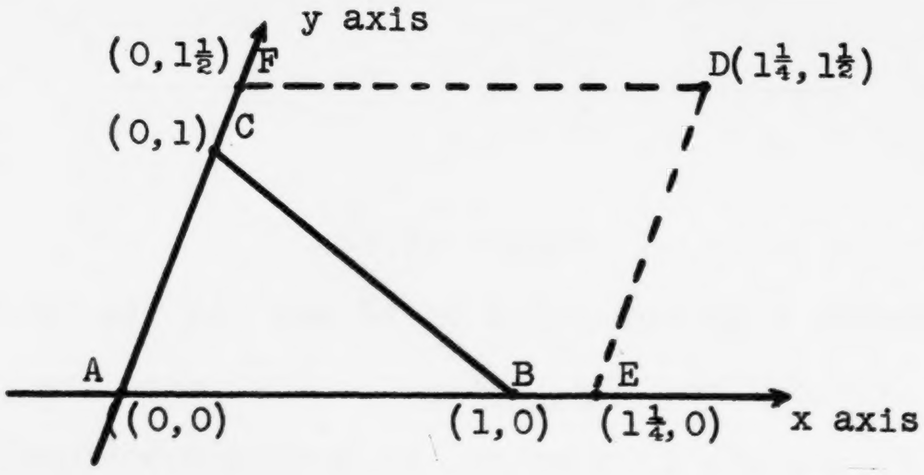


Figure 10.22

Let us go on to consider an isometry, g, of three non-collinear points A, B, C, and how to extend g to a fourth point in the plane of A, B, C.

Draw a triangle with plane coordinates as shown in Figure 10.22. On another paper trace ΔABC, calling it $\Delta A'B'C'$, and give A', B', C' the same coordinates respectively as A, B, C. Take any point D on the first paper and read its coordinates. Locate the point D' on the second paper with the same coordinates as D. Now place one paper over the other so that $A \longrightarrow A'$, $B \longrightarrow B'$, $C \longrightarrow C'$. Does $D \longrightarrow D'$? What conclusion seems indicated from this experiment? How can you extend g to other points of the plane?

10.11 Coordinates and Translations

As you will see, coordinates are quite useful in studying translations of points of a plane onto points of the same plane. Suppose point A has coordinates (1, 3) in some plane coordinate system and is mapped onto A', with coordinates (4, 5) by a translation. We can regard this translation as the composition of two motions. (See figure 10.23) The first moves a point 3 units in the direction of the positive x-axis and is followed by a second motion of 2 units in the direction of the positive y-axis. Any other point of the plane will also have an image under this composite translation.

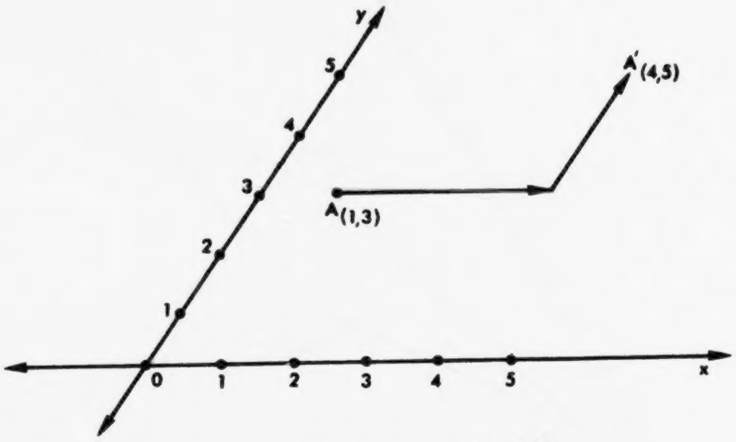


Figure 10.23

The rule of this translation is easy to write.

$$x \longrightarrow x + 3$$

$$y \longrightarrow y + 2$$

or simply $(x, y) \longrightarrow (x + 3, y + 2)$.

It is not hard to see that any mapping of the form

$$(x, y) \longrightarrow (x + a, y + b)$$

is a translation in a plane coordinate system.

In Chapter 9 we said that a translation in a plane maps lines onto parallel lines. Here, too, under the translation

$$(x, y) \xrightarrow{h} (x + 3, y + 2)$$

the point A (1, 3) [which means the point A with coordinates (1, 3)] maps onto A' (4, 5), and B (3, 8) maps onto B' (6, 10), so that $\overleftrightarrow{AB}$ maps onto the parallel line $\overleftrightarrow{A'B'}$ (See Figure 10.24).

Now consider the effect of the translation

$$(x, y) \xrightarrow{k} (x + 2, y + 5)$$

on the points A and A'. k maps A (1, 3) into B (3, 8), and it maps A' (4, 5) onto B' (6, 10), so that $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$.

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ and $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, the figure ABB'A' is a parallelogram. (It is a quadrilateral with opposite sides parallel.)

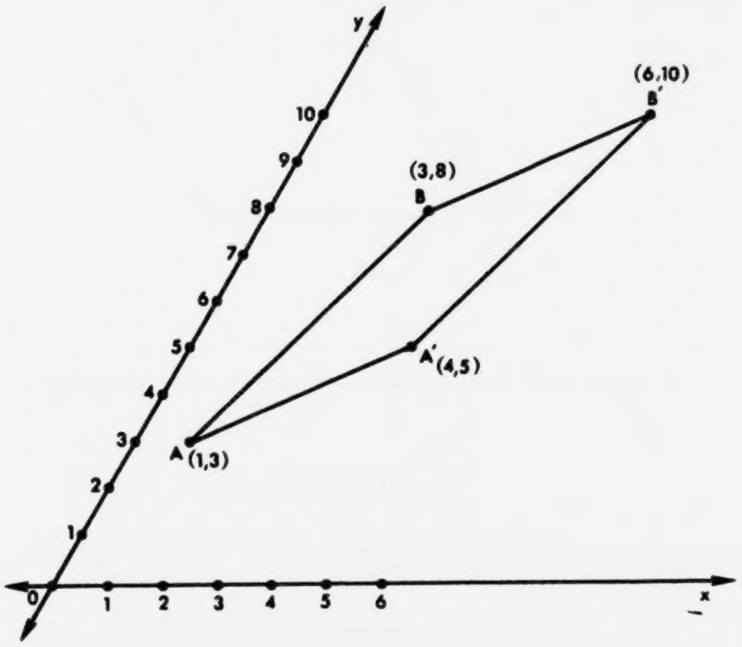


Figure 10.24

We can now check some facts about parallelograms in terms of coordinates, in particular, whether the diagonals bisect each other. But the coordinate formula for midpoints available to us is for line coordinates. We must therefore develop a formula for plane coordinates.

In Figure 10.25 we show only the diagonal $\overline{AB'}$ of parallelogram $ABB'A'$. Let M be the midpoint of $\overline{AB'}$, and consider the lines through A, M, and B that are parallel to the y-axis. These lines intersect the x-axis in points A_1, M_1, and B_1, respectively.

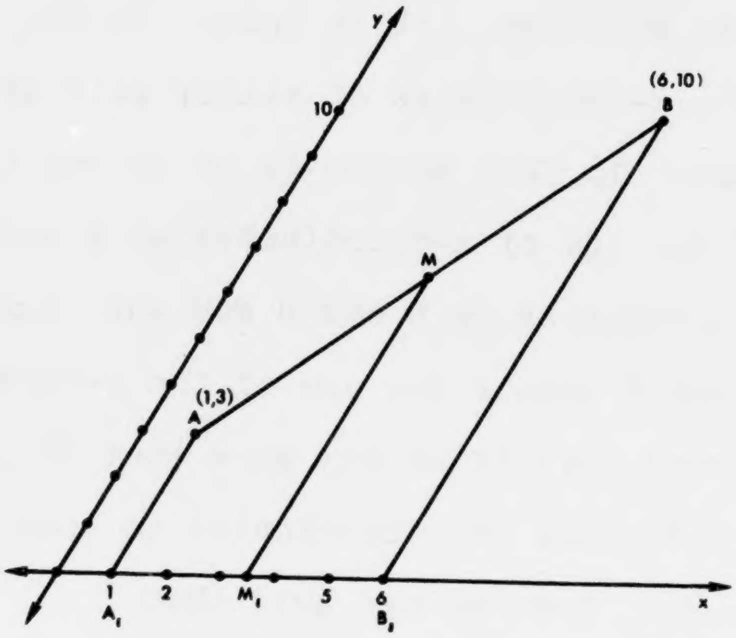


Figure 10.25

As you examine the coordinates of A_1, M_1, and B_1 in the line coordinate system on the x-axis, do you find that M_1 is the midpoint of $\overline{A_1B_1}$? Did you use the midpoint formula for line coordinate systems to check your answer? Since M acquires its x-coordinate from M_1, we conclude that the x-coordinate of M is also $\frac{1}{2}(1 + 6)$ or $\frac{7}{2}$. Using a diagram similar to 10.25 (drawing parallels to the x-axis), show that the y-coordinate of M is $\frac{1}{2}\cdot(3 + 10)$ or $\frac{13}{2}$.

In general, if P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2) then the midpoint of $\overline{PQ}$ has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Now verify that the coordinates of the midpoint of $\overline{BA'}$ are also $(\frac{7}{2}, \frac{13}{2})$. Does this verify that the diagonals of ABB'A' bisect each other?

There is a bonus in this development, which you will be asked to prove in an exercise. It is this: In any parallelogram the sum of the x-coordinates of either pair of opposite vertices is the same. In fact we can go on to say that ABCD is a parallelogram if the sum of x-coordinates of A and C equals the sum of the x-coordinates of B and D and the sum of the y-coordinates of A and C equals the sum of the y-coordinates of B and D. We can prove this if we can show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Let us start with ABCD and coordinates in some system as shown in Figure 10.26. Then we are told that

$$a + e = c + g \text{ and } b + f = d + h$$

It follows that

$$\frac{1}{2}(a + e) = \frac{1}{2}(c + g) \text{ and } \frac{1}{2}(b + f) = \frac{1}{2}(d + h).$$

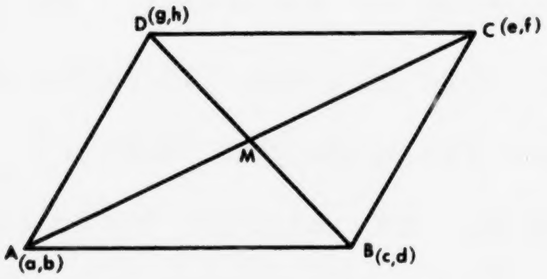


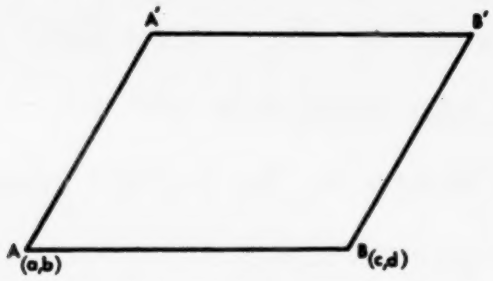
Figure 10.26

This means that $\overline{AC}$ and $\overline{BD}$ bisect each other, say in M. Thus M is the center of a point reflection that maps A onto C and B onto D.

In Chapter 9 we saw that a point reflection preserves parallelism. Hence, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. M is also the center of a point symmetry that maps A onto C and D onto B. Thus, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. We conclude that ABCD is a parallelogram.

10.12 Exercises

1. Let ABB' A' be a parallelogram. It can be regarded as having been formed by a translation under which $A \longrightarrow A'$ and $B \longrightarrow B'$. Suppose A and B have coordinate (a,b) and



(c,d) respectively in some coordinate system. Let the translation have the rule:

$$x \longrightarrow x + p \quad \text{and} \quad y \longrightarrow y + q.$$

Then A' has coordinates (a + p, b + q) and B' has coordinates (c + p, d + q).

(a) Using the midpoint formula show that $\overline{AB'}$ and $\overline{A'B}$ bisect each other.

(b) Show that the sum of the x-coordinates of A and B' is equal to the sum of the x-coordinates of A' and B.

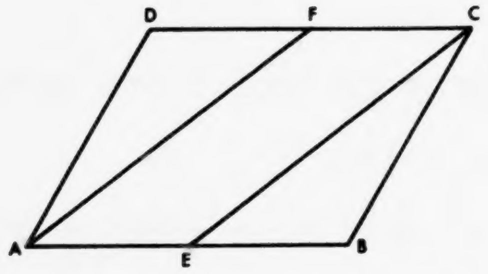
(c) Show that the sum of the y-coordinates of A and B' is equal to the sum of the y-coordinates of A' and B.

2. Suppose ABCD is a parallelogram and the coordinates of three

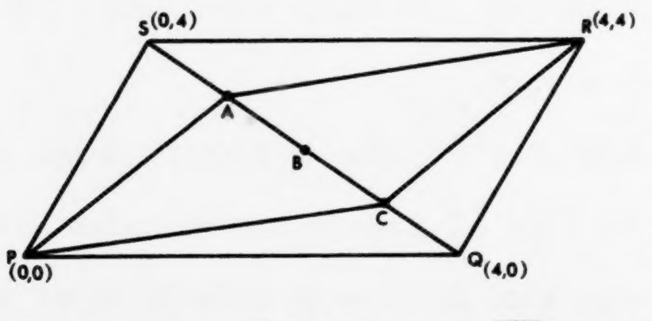
vertices are given. Find the coordinates of the missing vertex. Check your answers with a drawing.

(a)	A (0, 0)	B (3, 0)	D (0, 2)
(b)	A (0, 0)	B (3, 2)	D (2, 3)
(c)	A (2, 1)	B (5, 6)	C (0, 0)
(d)	A (3, 2)	C (-3, 2)	D (-2, 5)
(e)	B (-3, 2)	C (3, 3)	D (2, 5)
(f)	A (0, 0)	B (a, 0)	D (0, b)
*(g)	A (a, b)	B (c, d)	C (e, f)

3. Suppose ABCD is a parallelogram, that E is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{CD}$. Show that AECF is also a parallelogram. (You can simplify the proof by using the coordinate system in which A, B, D have coordinates (0, 0), (1, 0) and (0, 1) respectively.)



4. (a) Using the indicated coordinates, show that PQRS is a parallelogram.



(b) Suppose B is the midpoint of $\overline{SQ}$, that A is the midpoint of $\overline{SB}$ and C is the midpoint of $\overline{BQ}$. Show that PCRA is also a parallelogram.

5. For the parallelogram PQRS in Exercise 4 take any suitable coordinates for the vertices and show again that PCRA is a parallelogram. What is the significance of taking any suitable coordinates for P, Q, R, S?
6. Using coordinates, show that translations preserve midpoints.

10.13 Perpendicular Lines

In Chapter 9 we studied reflections in a line. In this section we use such reflections to review and extend the idea of perpendicular lines.

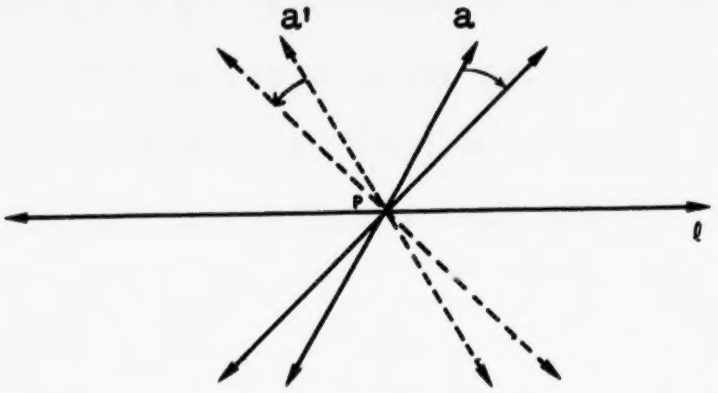


Figure 10.27

In the diagram of Figure 10.27 you see that the reflection of line a in line ℓ is line a'. Now a and a' are different lines, and they intersect each other at point P. Why must P be a point of ℓ? Imagine that a rotates around P as a pivot in the clockwise direction. Let a' continue to be the reflection of a. How does a' rotate? In the course of rotation, does a' ever become the same as a?

Now rotate a in a counterclockwise direction. In the course of this rotation does a' again become the same as a?

We see that a can be its own image, as it rotates about P, in two ways. In one of these $a = \ell$; in the other $a \neq \ell$. In general two lines are perpendicular if they are different lines, and one of them is its own image under a line reflection in the other.

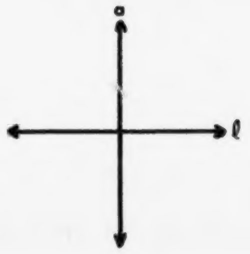


Figure 10.28

We denote that a is perpendicular to ℓ by writing $a \perp \ell$. Note that ℓ is also its own image under a reflection in a (Figure 10.28). So $\ell \perp a$ whenever $a \perp \ell$. Also note that the plane is separated by each of the two perpendicular lines into two half-planes.

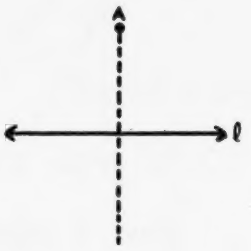
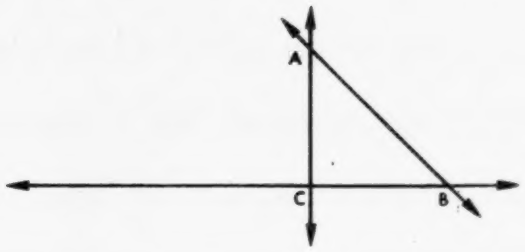


Figure 10.29

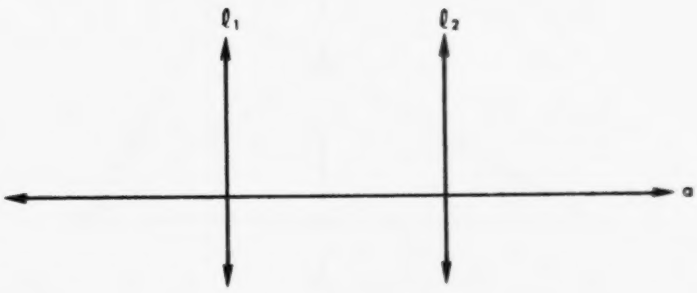
On a piece of paper draw line ℓ and mark a point A, either on or off ℓ, as in Figure 10.29. Fold the paper along a line containing A such that one part of ℓ falls along the other. In how many ways can this fold be made? You know that the line of the crease is perpendicular to ℓ. It would seem then that in a given plane there is exactly one line containing a given point that is perpendicular to a given line.

10.14 Exercises

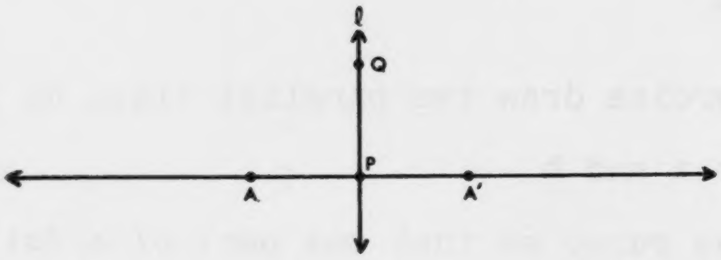
1. For this exercise draw two parallel lines on your paper, calling them a and b.
   - (a) Fold the paper so that one part of a falls along the other part. Label the crease c. Is $c \perp a$? Why?
   - (b) For the fold you made in (a), does part of b fall along another part of itself? What bearing does your answer have on the perpendicularity relation of c and b?
   - (c) Tell how the results of this experiment support or do not support this statement: If two lines are parallel, a line perpendicular to one is perpendicular to the other.



2. Suppose, as shown in the diagram above, that $\overleftrightarrow{AC} \perp \overleftrightarrow{BC}$. Can $\overleftrightarrow{AB}$ also be perpendicular to $\overleftrightarrow{BC}$? Be ready to support your answer.



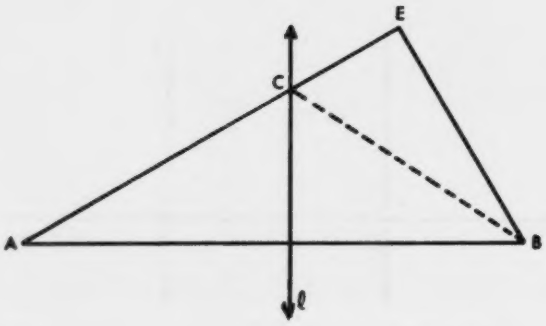
3. Suppose, as shown in the diagram above, that $\ell_1 \perp a$ and $\ell_2 \perp a$. Can ℓ_1 intersect ℓ_2? Be ready to support your answer. If they do not intersect, how do you describe their relationship?



4. Let A' be the image of A under a reflection in ℓ, as shown in the diagram above, and let $\overleftrightarrow{AA'}$ intersect ℓ in P. What is the image of P under this reflection? You know that a reflection in a line preserves distance. Compare AP with A'P. We see that $\ell \perp \overleftrightarrow{AA'}$ and P is the midpoint of $\overline{AA'}$. We call ℓ the midperpendicular or perpendicular bisector of $\overline{AA'}$. Show that every point in ℓ is as far from A as from A'. We can state the result of this exercise as follows: Every point in the midperpendicular of a line segment is as far from one endpoint of the segment as from the other.

5. Suppose ℓ is the midperpendicular of $\overline{AB}$. Suppose E is in the B-side of ℓ, as shown in the diagram below.

   (a) We can show that EA > EB as follows: (You are to give a reason for each statement.)



(1) A and B are on opposite sides of ℓ.

(2) E and A are on opposite sides of ℓ.

(3) $\overline{EA}$ intersects ℓ in a point, say C, which is between A and E.

(4) $EA = EC + CA$

(5) $EC + CB > EB$

(6) $CB = CA$

(7) $EC + CA > EB$

(8) $EA > EB$

(b) Suppose F is in the A-side of ℓ. Show by an argument like the one in (a) that $FB > FA$.

(c) State in words the proposition that was proved in (a) and (b).

10. 15 Using Coordinates for Line and Point Reflections

For our present purpose we use a special coordinate system in which the axes are perpendicular lines. Such special coordinate systems are called rectangular coordinate systems. We shall study reflections in their axes. Let ℓ_x be the line reflection in the x-axis and let ℓ_y be the line reflection in the y-axis. Let P have coordinates (2, 3).

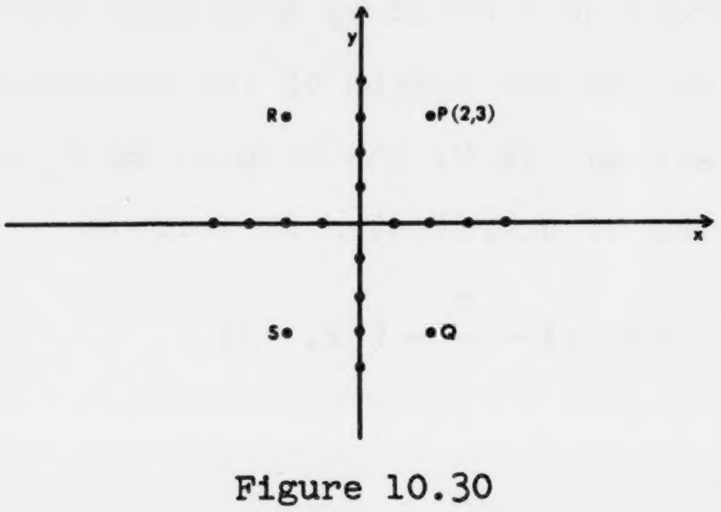


Figure 10.30

If P $\xrightarrow{\ell_x}$ Q, what are the coordinates of Q?

If P $\xrightarrow{\ell_y}$ R, what are the coordinates of R?

If $Q \xrightarrow{\ell_y} S$, what are the coordinates of S?

We can form the composition $\ell_y \circ \ell_x$ by taking the reflection in the x-axis, followed by the reflection in the y-axis. What is the image of P under this composition? Does the image of P change if we reverse the order of the reflections?

Now let us consider the same questions for a point A with coordinates (a, b).

If $A \xrightarrow{\ell_x} B$, what are the coordinates of B?

If $A \xrightarrow{\ell_y} C$, what are the coordinates of C?

If $A \xrightarrow{\ell_y \circ \ell_x} D$, what are the coordinates of D?

Do you agree that the rules for ℓ_x and ℓ_y, when given in forms of coordinates of points are as follows:

for ℓ_x: $x \longrightarrow x$, $y \longrightarrow -y$ or $(x, y) \longrightarrow (x, -y)$

for ℓ_y: $x \longrightarrow -x$, $y \longrightarrow y$ or $(x, y) \longrightarrow (-x, y)$

for $\ell_y \circ \ell_x$: $x \longrightarrow -x$, $y \longrightarrow -y$ or $(x, y) \longrightarrow (-x, -y)$

You must surely have noted by this time that the $\ell_x \circ \ell_y$ is a point reflection in the origin of the coordinate system. If we denote this reflection in O, the origin, as P_o we can state the rule of P_o in terms of coordinates as follows:

$$(x, y) \xrightarrow{P_o} (-x, -y)$$

10.16 Exercises

1. For each of the points with coordinates in a rectangular

coordinate system given below find the coordinates of its image:

(1) under the line reflection in the x-axis

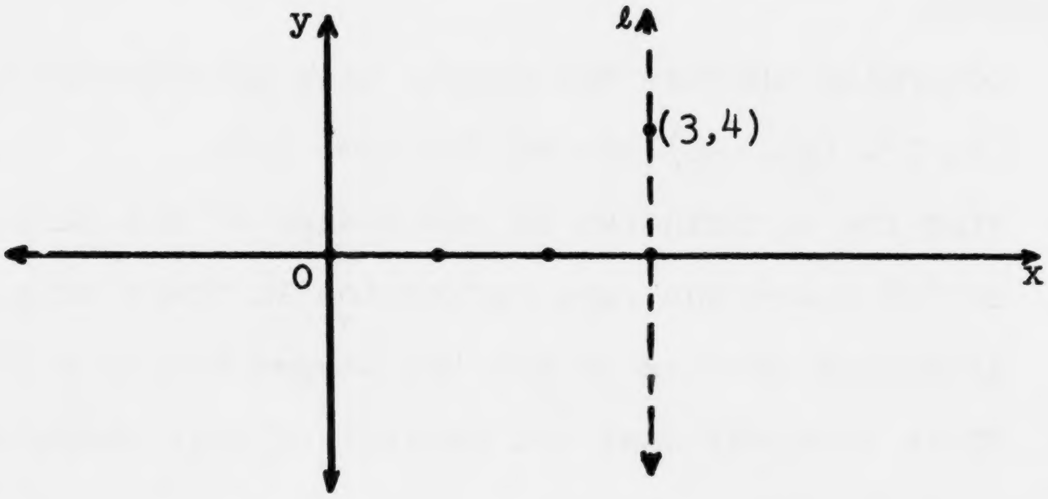
(2) under the line reflection in the y-axis, and

(3) under the point reflection in the origin

(a) (3, 5) (c) (5, -3) (e) (2, 0) (g) (-3, -1)

(b) (-3, 5) (d) (-3, -5) (f) (0, 5) (h) (82, -643)

2. Let ℓ be the line that is perpendicular to the x-axis containing the point with coordinates (3, 4) in some rectangular coordinate system. Let points have the coordinates listed below. Find the coordinates of the image of each point under a line reflection in ℓ.



(a) (1, 4) (c) (3, 2) (e) (0, 0) (g) (8, -3)

(b) (0, 3) (d) (-3, -1) (f) (10, 0) (h) (x, y)

3. Let m be the line that is perpendicular to the y-axis of a rectangular coordinate system, and contains the point with coordinates (3, 4). Find the coordinates of the image of each point in Exercise 2 under a line reflection in m.

4. Find the coordinates of the image of each point in Exercise

2 under a point reflection in the origin O.

5. Let A and B have rectangular coordinates (1, 5) and (3, 1) respectively.
   (a) Let $A \xrightarrow{\ell_x} A'$ and $B \xrightarrow{\ell_x} B'$. Find the coordinates of A' and B'.
   (b) Find the coordinates of the midpoint M of $\overline{AB}$ and let $M \xrightarrow{\ell_x} M'$. Find the coordinates of M'.
   (c) Show that M' is the midpoint of $\overline{A'B'}$.
6. Show that the line reflection in the x-axis preserves midpoints. You might wish to work with points A and B having coordinates (2a, 2b) and (2c, 2d).
7. Show that the point reflection in the origin O preserves midpoints.
8. (a) Determine whether the points with coordinates (1, 3), (4, 1), (10, -3) are on the same line.
   (b) Find the coordinates of the images of the three points in (a) under the line reflection in the x-axis, and determine whether or not the images are on a line.
   (c) State in words what the results of this exercise seem to indicate.
9. Using the three points in Exercise 8 show that their images under a point reflection in the origin are on a line.

10.17 What is an Angle?

No doubt the word "angle" has some meaning for you. However, you may find it quite difficult to describe it precisely. To see just how difficult, you might try to explain what an angle is to a youngster in the first or second grade. A particularly difficult task would be to describe it without diagrams.

(To see how important angles are in everyday thinking, one can look up the word angle and related words in the dictionary. You will be asked to do this in an exercise.)

You probably would say that the diagram in Figure 10.31 represents an angle. But is the entire angle shown? Is the fact that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ have a common endpoint significant? Are the points between A and B part of the angle? These are some of the questions that must be answered in giving a precise mathematical meaning to the word "angle."

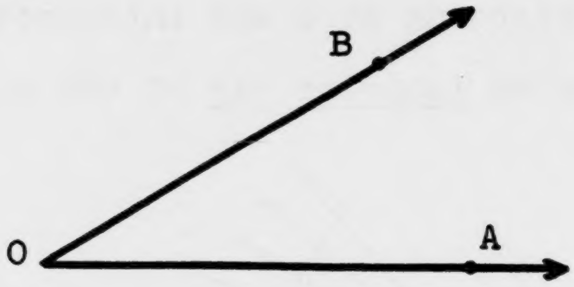


Figure 10.31

After carefully reading the following you should be able to answer all of them.

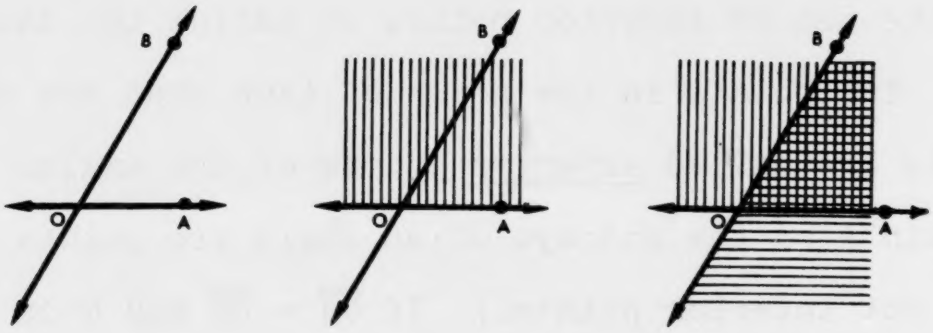


Figure 10.32

Let us start with two lines intersecting at O, as shown in Figure 10.32. We name them $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$. With these lines given we shall show in stages how the angle emerges. First we take the halfplane with boundary $\overleftrightarrow{OA}$ that contains B. It is indicated by vertical shading lines. Then we take the halfplane with boundary $\overleftrightarrow{OB}$ that contains A. It is indicated by horizontal shading lines. The region that is crosshatched is the angle. It is the intersection of the two halfplanes. It is named $\angle AOB$. Each point used in the name signifies something. O is the point of intersection of the two lines. It is called the *vertex* of the angle. A and B tell us which halfplane to take. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the *endrays* or *sides* of the angle. There are other rays in the angle. Any ray starting at O and intersecting any interior point of $\overline{AB}$ is called an *interior ray* of the angle. All points of the angle, not in endrays, are called *interior* points of the angle and the set of interior points is called the interior of the angle. The points in the plane of $\angle AOB$ that are not points of the angle are called *exterior* points of the angle. (Note that the points on the endrays of an angle are points of the angle, but not interior points.) If $\overleftrightarrow{OA} = \overleftrightarrow{OB}$ and O is between A and B, then we cannot build up the angle as described above.

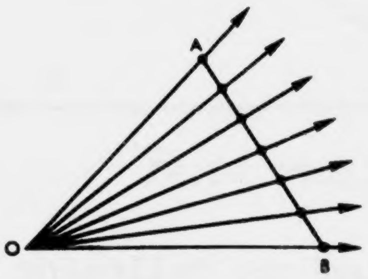


Figure 10.33

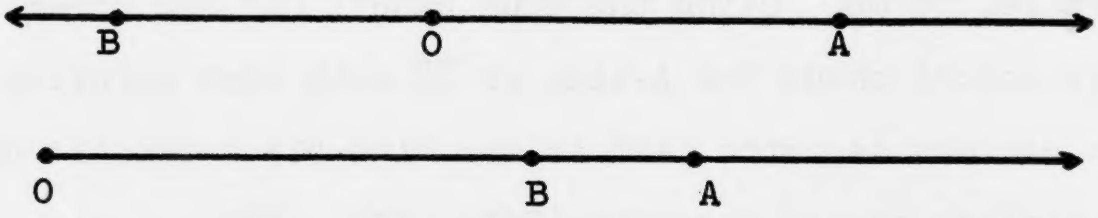


Figure 10.34

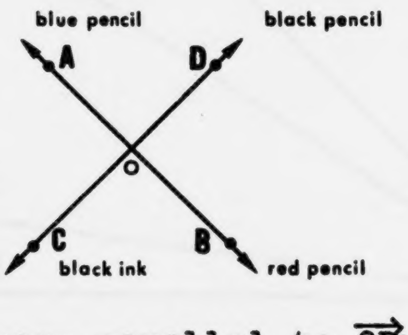
Nevertheless we call any halfplane with boundary $\overleftrightarrow{AB}$, with O as vertex, a straight angle. If O is not between A and B, then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ name the same ray. In this case $\angle AOB$ collapses into a ray, and we will call $\angle AOB = \overrightarrow{OA} = \overrightarrow{OB}$ a zero angle.

Does our definition of an angle differ from what you have previously learned about angles?

If so, we ask you to consider the fact that a definition is an agreement among ourselves as to what a word shall mean. Once the agreement is made, however, we must stick with it and with its consequences.

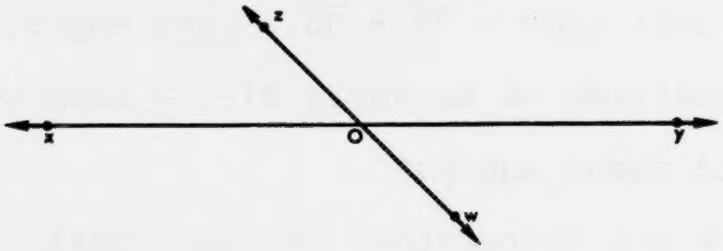
10.18 Exercises

1. Draw two intersecting lines on your paper and label points as in the diagram. Using ordinary black pencil shade the

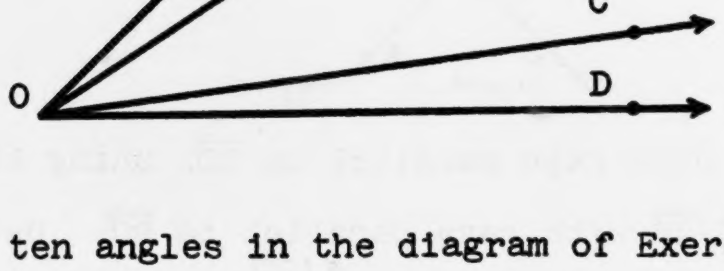


D-side of $\overleftrightarrow{AB}$ with rays parallel to $\overrightarrow{OD}$; using black ink shade the C-side of $\overleftrightarrow{AB}$ with rays parallel to $\overrightarrow{OC}$. Using red pencil (or any available color) shade the B-side of $\overleftrightarrow{CD}$ with rays parallel to $\overrightarrow{OB}$. Using the blue pencil (or any other available color) shade the A-side of $\overleftrightarrow{CD}$ with rays parallel to $\overrightarrow{OA}$. You can now describe $\angle AOD$ as the blue-black pencil angle. In similar manner describe $\angle BOD$, $\angle AOC$, $\angle BOC$.

2. Using the diagram shown below, name:
   (a) four straight angles
   (b) four zero angles
   (c) four other angles



3. Using the diagram shown below, describe as a single angle, if possible:
   (a) $\angle AOB \cup \angle BOC$
   (b) $\angle AOC \cap \angle COB$
   (c) $\angle AOC \cup \angle BOD$
   (d) $\angle AOC \cap \angle COD$



4. There are ten angles in the diagram of Exercise 3. Four of

them are zero angles. Name the other six.

5. You may have noticed that there are many resemblences between an angle and a segment. For each sentence below about segments write one that resembles it and is about angles.
   - (a) A segment has two endpoints.
   - (b) A segment is a set of points.
   - (c) The interior of a segment contains points of a segment other than its endpoints.
   - (d) If C and D are interior points of $\overline{AB}$, then every point in $\overline{CD}$ is in $\overline{AB}$.
6. Consult a dictionary to find five uses of angles.

10.19 Measuring an Angle

You have noted in Exercise 5 above a number of resemblences between angles and segments. It should not surprise you that the measurement of angles also resembles the measurement of segments. To measure a segment we use a scaled ruler. To measure an angle we use a scaled protractor. The numbers on a ruler are assigned to points. The numbers on a protractor are assigned to rays. (In Figure 10.35 only three rays are shown).

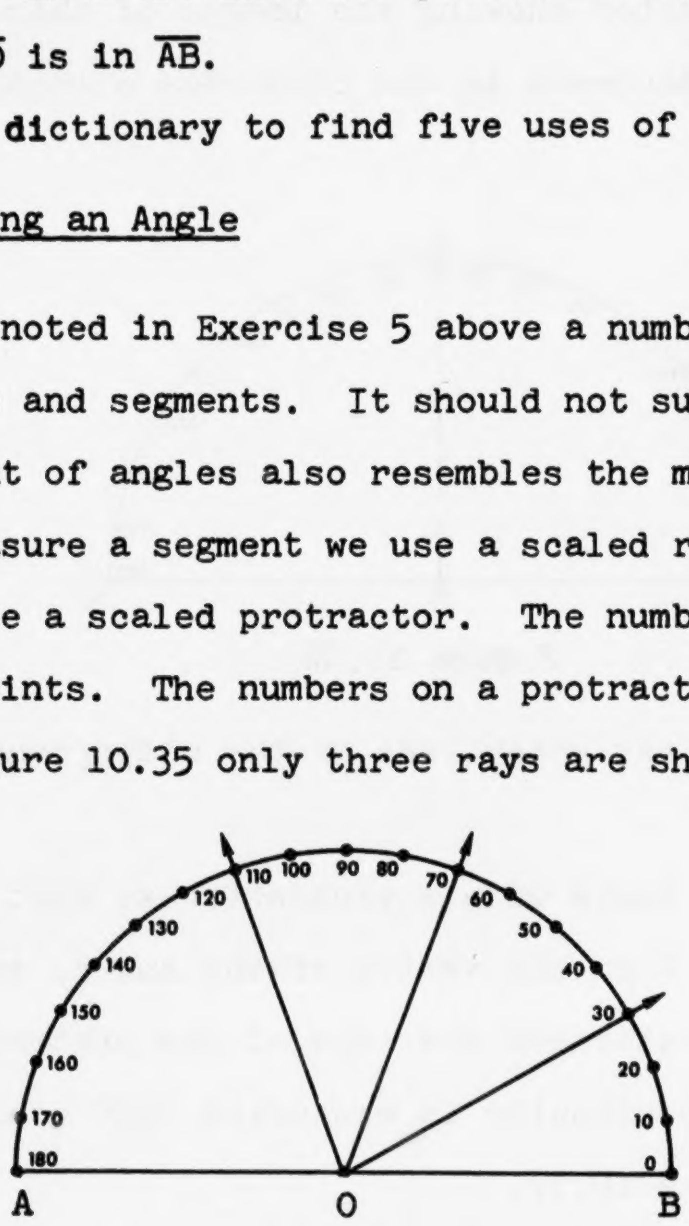


Figure 10.35

Numbers on ordinary rulers start at zero and go on as far as permitted by the scale unit and the length of the ruler. No matter how large the protractor we are going to use, its numbers start with 0 and end with 180.

As you see, a protractor has the shape of a semi-circle. $\overline{AB}$ is the diameter of the protractor and O is its center. In Figure 10.35 the numbers increase in the counter-clockwise direction. However, if we reflect the protractor in the line that is the midperpendicular of $\overline{AB}$, then each number n is mapped onto 180-n. In a protractor showing the images of this line reflection, the numbers increase in the clockwise direction (Figure 10.36).

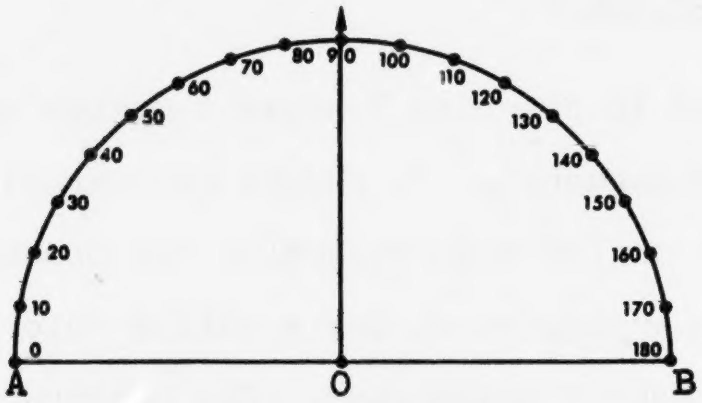


Figure 10.36

In either case the ray which lies in the midperpendicular of $\overline{AB}$ is assigned 90.

To measure an angle with a protractor we must begin by placing the center O on the vertex of the angle, and each ray of the angle must intersect the edge of the protractor. Perhaps the position of a protractor in measuring ∠ABC could be like that shown in Figure 10.37.

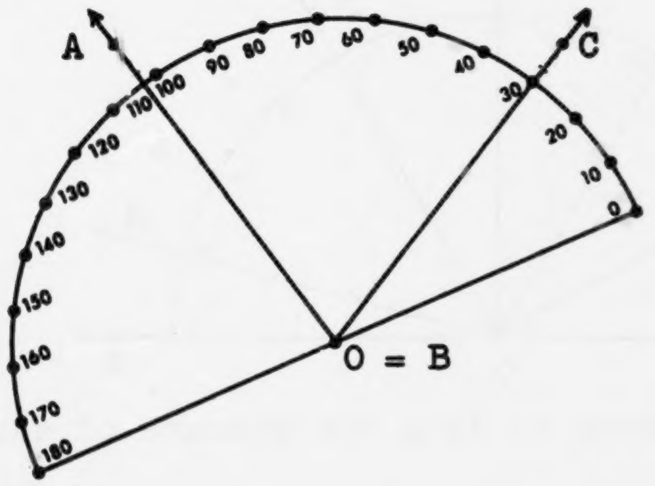
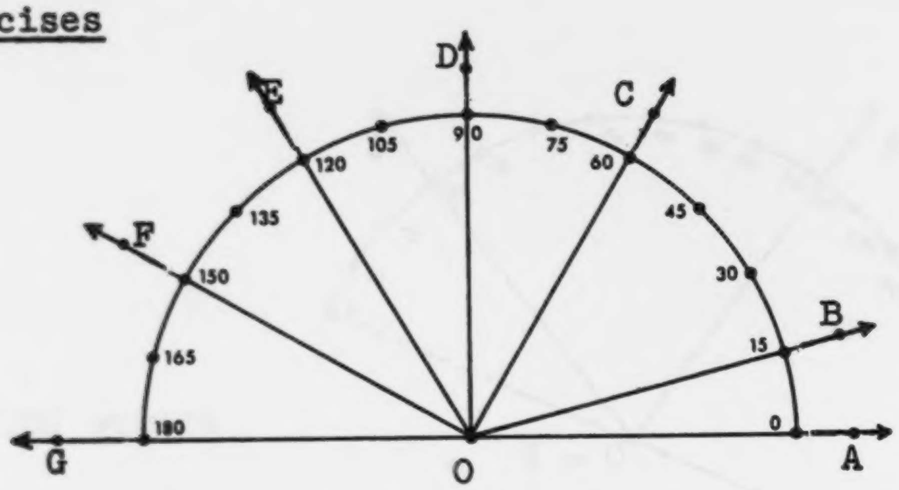


Figure 10.37

In this position the protractor assigns the number 30 to $\overrightarrow{BC}$ and 103 to $\overrightarrow{BA}$. It cannot come to you as a surprise that the measure of $\angle ABC$ is 103 - 30 or 73. Or if you computed 30 - 103, you would then take the absolute value of the difference, just as we did in measuring line segments. When the protractor is graduated from 0 to 180 we call the unit of measurement a degree. When we say that the measure of $\angle ABC$ is 73 degrees, or 73°, we are also saying that we used a protractor graduated from 0 to 180. (There are other types of protractors graduated from 0 to other numbers.) In measuring a line segment we like to place the ruler so that it assigns 0 to one end, for this considerably simplifies the computation. In measuring an angle we also like to place the protractor so that zero is assigned to an endray, for the same reason.

The abbreviation for "degree measure of $\angle ABC$" is $m\angle ABC$.

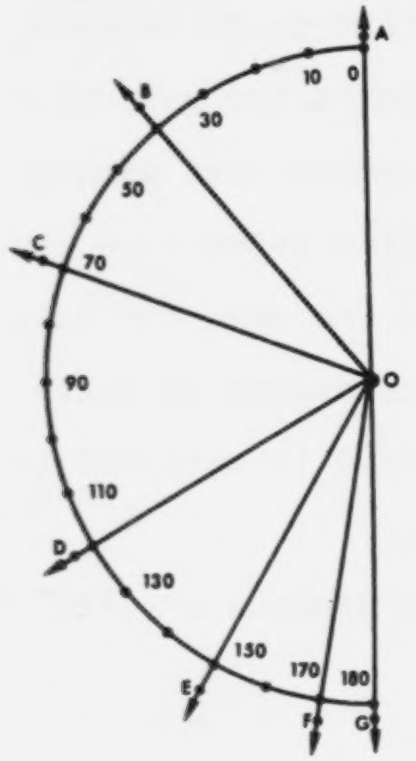
10.20 Exercises


D
E
C
F
B
G
O
A
0
15
30
45
60
75
90
105
120
135
150
165
180

1. Consult the diagram above to find the measure of each angle listed below:

   (a) ∠AOC
   (b) ∠BOC
   (c) ∠COB
   (d) ∠AOF
   (e) ∠BOE
   (f) ∠FOB
   (g) ∠GOC
   (h) ∠EOE
   (i) ∠GOA
   (j) ∠AOG
   (k) ∠AOD
   (l) ∠DOG

2. Using the diagram shown below, find the measure of each angle listed below:


A
B
C
D
E
F
G
O
0
10
30
50
70
90
110
130
150
170
180

   (a) ∠AOC
   (b) ∠BOD
   (c) ∠DOC
   (d) ∠FOG
   (e) ∠GOE
   (f) ∠FOB

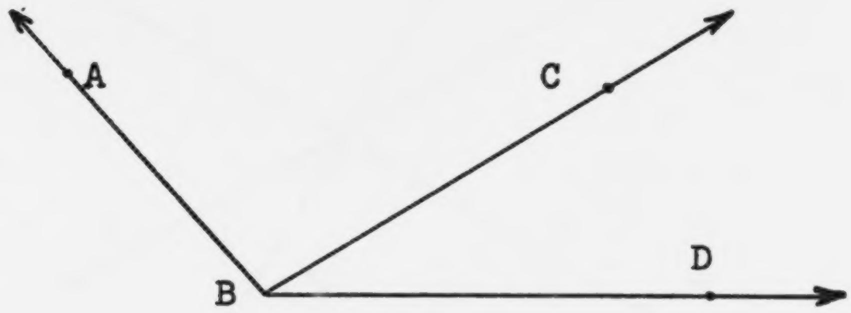
3. Consult the diagram of Exercise 2 to compute each of the following:

   (a) $m\angle AOB + m\angle BOC$

   (b) $m\angle GOA - m\angle COA$

   (c) $2m\angle AOB + 3m\angle COD$

4. If two angles in a plane have the same vertex, and only one ray in common, they are called a pair of adjacent angles. From the diagram determine which pair of angles listed below are adjacent angles.

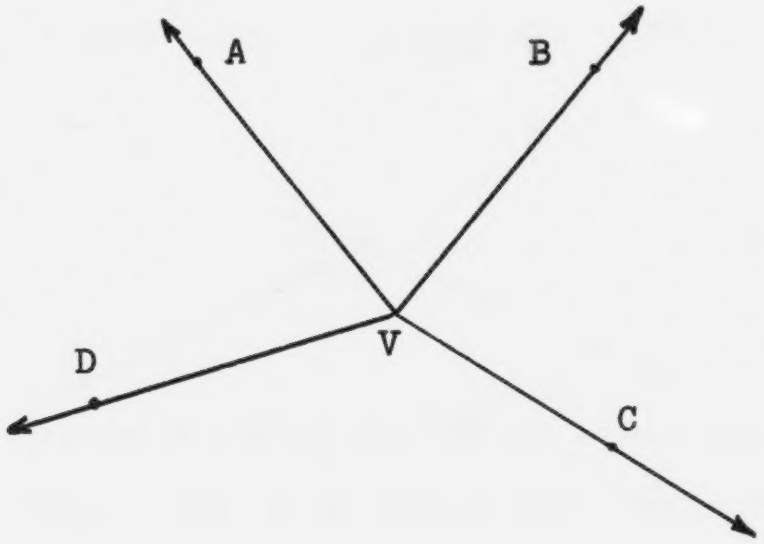


   (a) $\angle ABD$ and $\angle CBD$

   (b) $\angle ABC$ and $\angle CBD$

   (c) $\angle DBA$ and $\angle ABC$

5.

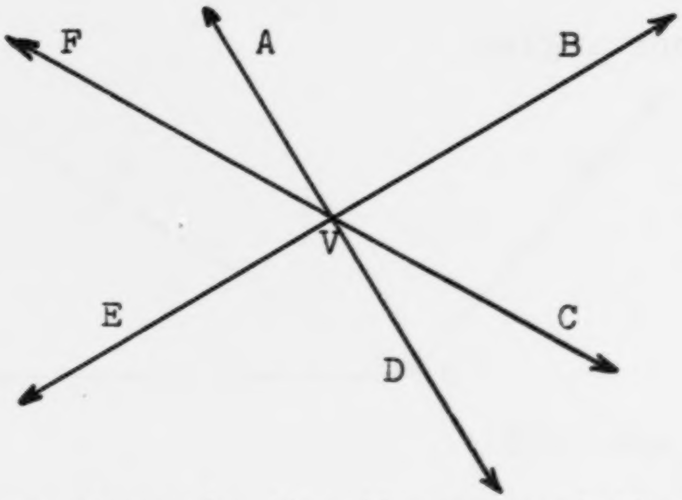


In the diagram above name as many pairs of adjacent angles as you can.

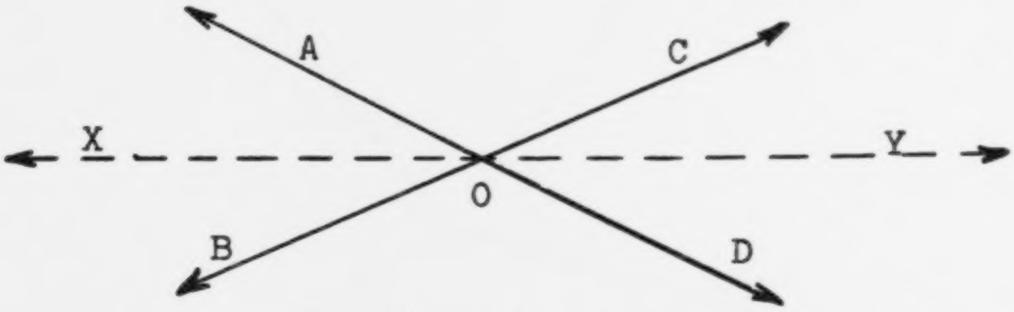
6. Using an illustration show that the sum of the measures of two adjacent angles is not necessarily the measure of an angle.

7. Using a protractor, find the measure of each of the angles listed for the diagram below:

(a) $\angle AVB$	(d) $\angle EVC$	(g) $\angle BVF$
(b) $\angle DVC$	(e) $\angle AVF$	(h) $\angle AVD$
(c) $\angle AVC$	(f) $\angle FVD$	



#8. Consider $\angle AOB$, as shown in the diagram and the point reflection of $\angle AOB$ in vertex O. Under this reflection the image of endray $\overrightarrow{OA}$ is $\overrightarrow{OD}$, the opposite ray. What is the image of



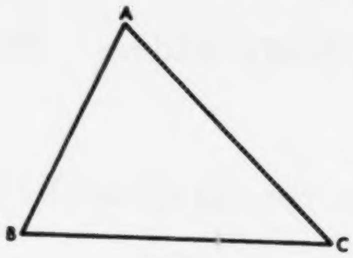
$\overrightarrow{OB}$? What is the image of $\overrightarrow{OX}$, an interior ray of $\angle AOB$? What is the image of $\angle AOB$? The image of an angle under a point reflection in its vertex is its vertical angle.

9. (a) In the diagram of Exercise 8, what is the vertical angle of $\angle DOC$?

   (b) What is the vertical angle of $\angle AOB$?

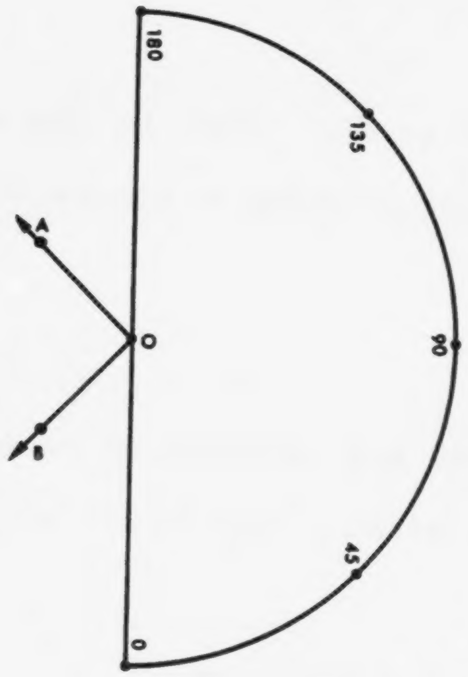
10. Using a protractor show that the measure of an angle is equal to the measure of its vertical angle.

11. $\overline{AB}$ and $\overline{AC}$ are two sides of a triangle. They determine two endrays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ of an angle. In this sense every triangle has three angles. We can name them $\angle A$, $\angle B$, and $\angle C$.



Measure each angle of the triangle and then find the sum of their measures.

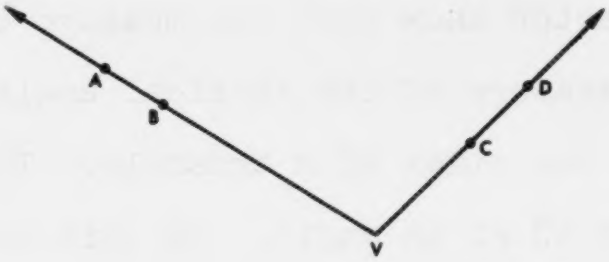
12.



(a) Explain why we cannot use the protractor in the position shown here to measure $\angle AOB$.

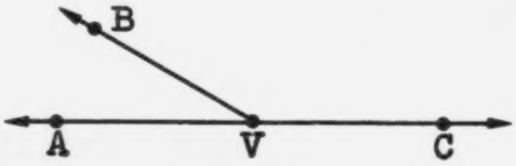
(b) Can the measure of an angle be greater than 180°? Explain your answer.

13. Look at ∠BVC in the diagram below. Now look at ∠AVD. Compare their measures. (Try to answer without the use of a protractor.)



14. You know that two perpendicular lines determine four angles disjoint except for their sides. What is the measure of each angle?

15. (a) Measure ∠AVB in the diagram below. Using your result, find the measure of ∠BVC.



(b) Suppose the measure of ∠AVB is 40. What is the measure of ∠BVC? Try to answer without using a protractor.

10.21 Boxing the Compass

As you know the marks on a ruler are located by repeated bisections, once we start with inch marks. The first bisection produces a ruler like this:

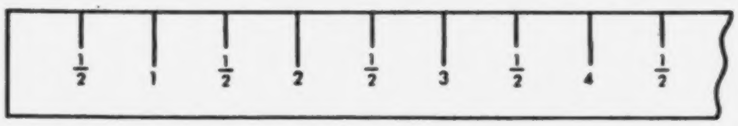


Figure 10.38

A second bisection produces a ruler like this.

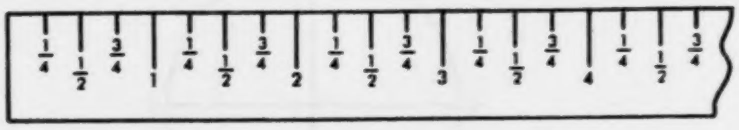
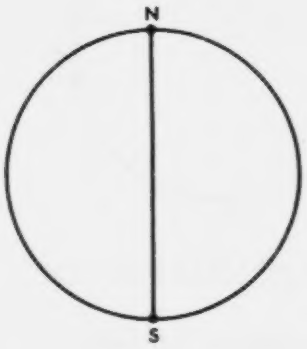


Figure 10.39

Repeated bisections produce eighths, sixteenths, and thirty-seconds.

There is an analogous situation for protractors, more accurately for two protractors, placed diameter to diameter to form a circle. It is called boxing the compass, and gives the type of compass used in certain types of marine navigation.

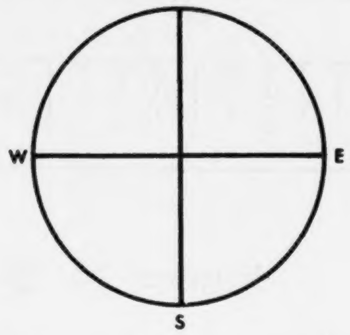
A diameter of either protractor bisects the circle. One end of this diameter is marked N (north) and the other is marked S (south). (Figure 10.40)



First Bisection

Figure 10.40

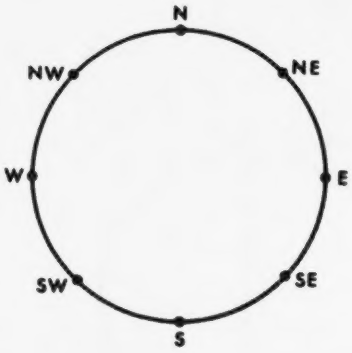
Bisecting each semi-circle locates E (east) and W (west). (Figure 10.41)



Second Bisection

Figure 10.41

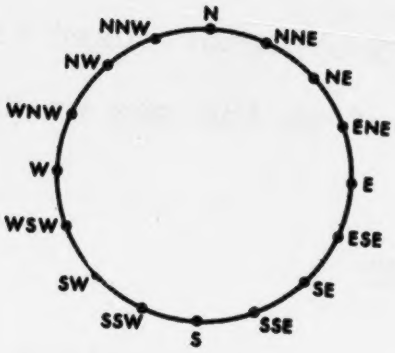
Bisecting each of the four arcs locates NE (northeast), SE (southeast), SW (southwest), and NW (northwest). Notice we do not say "eastnorth." The rule is that "north" takes precedence over "east" and "west" because it appeared earlier in the process. Likewise, we say southeast because "south" appears before "east" in the process.



Third Bisection

Figure 10.42

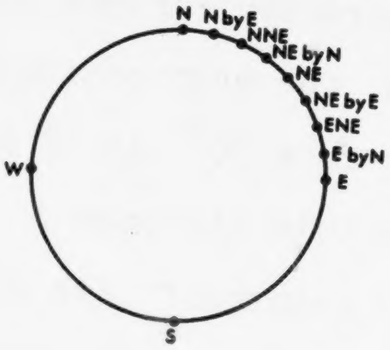
Bisecting each of the eight arcs locates NNE (northnortheast), ENE, ESE, SSE, etc. In the designation NNE, N appears before NE because it is on the N side of NE. Thus, ENE is on the E side of NE.


N
NNE
NE
ENE
E
ESE
SE
SSE
S
SSW
SW
WSW
W
WNW
NW
NNW

Fourth Bisection

Figure 10.43

The fifth bisection completes boxing of the compass. The midpoint of the arc between N and NNE is called N by E (north by east); the one between NNE and NE is called NE by N. Not NNE by S. Why not?


N
N by E
NNE
NE by N
NE
NE by E
ENE
E by N
E
W
S

Fifth Bisection

Figure 10.44

Make a complete diagram showing the compass "boxed."

The circle is now subdivided into 32 arcs, each having the same measure. The mariner calls each measure a "point." This point does not mean the point we study in geometry. The terms "half-point" and "quarter-point" describe still smaller arc lengths. Since there are 8 points to one quarter of a circle, one point corresponds to $11\frac{1}{4}^{\circ}$. So a change of course of one-quarter point corresponds to a change of approximately 3°.

Thus the kind of "protractor" used in some types of navigation is quite different from the one we described in Section 10.19.

10.22 More About Angles

Draw ray $\overrightarrow{VA}$ on your paper and place your protractor so that $\overrightarrow{VA}$ is assigned zero. In how many possible positions can you hold the protractor? (Were you careful to place the center of the protractor on V?) For each position, draw a ray, starting at V, to which the protractor assigns the number 70. How many such rays can you draw for each position? How many angles then can you draw having measure 70° if $\overrightarrow{VA}$ is one of the sides?

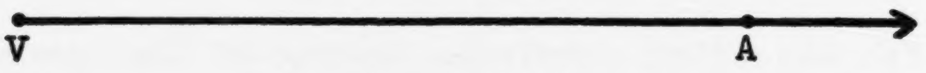


Figure 10.45

Do you agree with this statement?

For each ray, for each halfplane containing this ray in its boundary, and for each number $\underline{x}$, such that $0 \leq x \leq 180$, there is exactly one angle with measure $\underline{x}$ that has the given ray as one side.

This statement is going to be very useful to us in our study of angles. For instance, we can now show that any angle, such as $\angle AVB$, can be divided into two angles that have equal measures. To do this, we place a protractor in the position

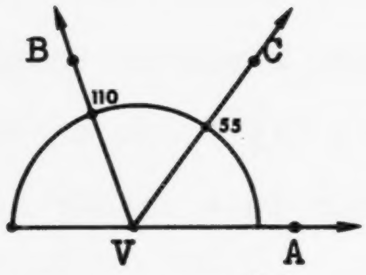


Figure 10.46

shown; see that 110 is assigned to $\overrightarrow{VB}$ and reason that we are looking for the ray that is assigned $\frac{1}{2}$ of 110 or 55. We look for 55 on the protractor and draw $\overrightarrow{VC}$, the ray that is assigned 55. What is $m\angle BVC$? $m\angle CVA$? Have we divided $\angle AVB$ into two angles as claimed? How can we use the statement above to show that an angle has exactly one midray?

In our example $\overrightarrow{VC}$ is called the midray of $\angle AVB$ for obvious reasons; it bisects the angle, and is therefore also called the bisector of $\angle AVB$. Explain why any angle, other than a straight angle, has only one midray.

We pause here to introduce some terms describing angles. If the measure of an angle is 90, it is called a right angle. If the measure of an angle is between 0 and 90, it is called an acute angle. If the measure of an angle is between 90 and 180, it is called an obtuse angle.

10.23 Exercises

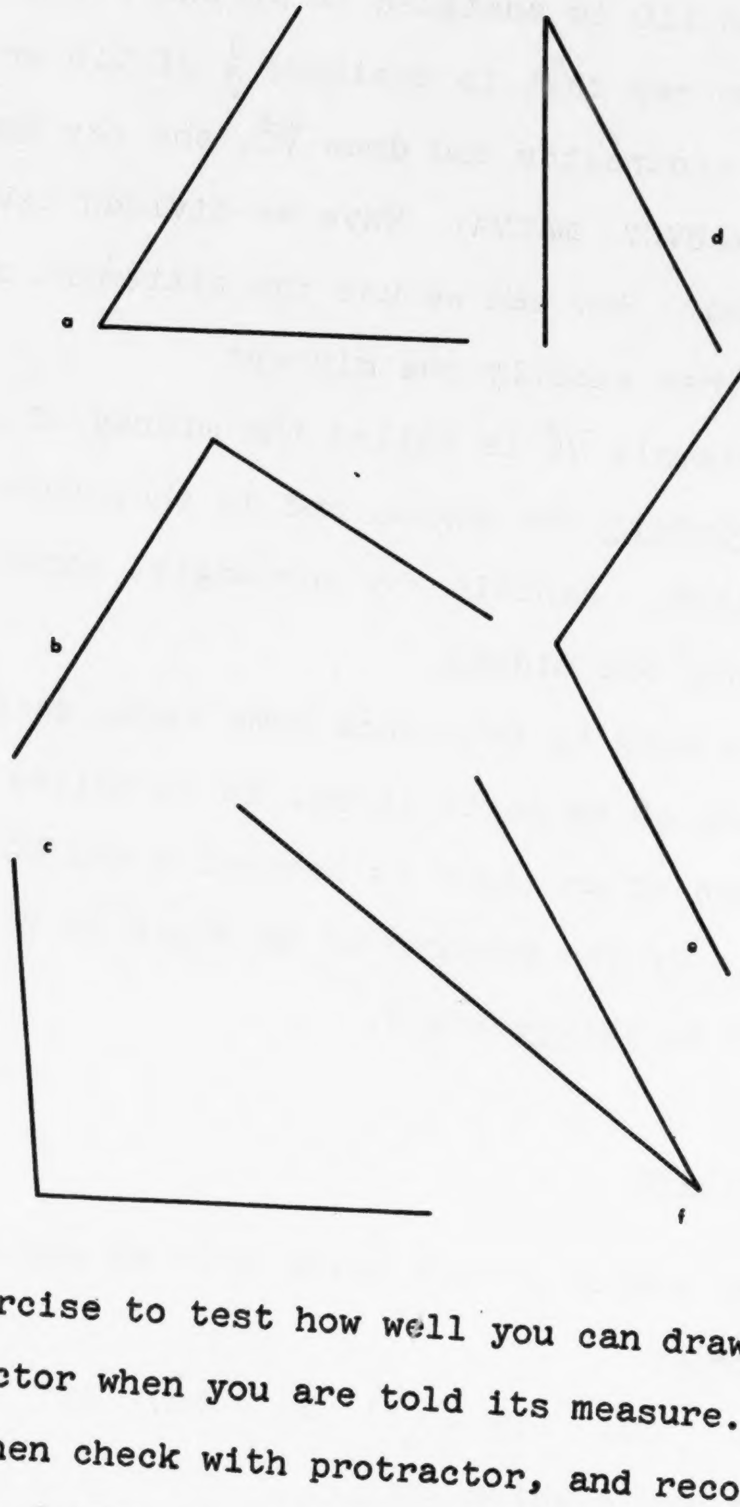
1. For each number listed below draw an angle whose measure is that number:

   (a) 35 (b) 135 (c) 18 (d) 90 (e) 180 (f) 0

2. Draw an angle which is:

   (a) a right angle
   (b) an acute angle
   (c) an obtuse angle

3. This exercise is a test of how well you can estimate the measure of an angle from a diagram. For each of the angles given, estimate the measure, record your estimate, and then use your protractor to check your estimate.



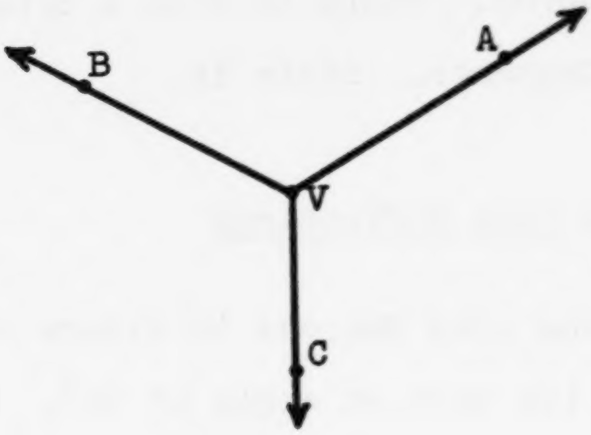
4. This is an exercise to test how well you can draw an angle without protractor when you are told its measure. Draw the angle first, then check with protractor, and record the error for each of the following measurements:

(a) 45°
(b) 30°
(c) 150°
(d) 90°
(e) 60°
(f) 120°

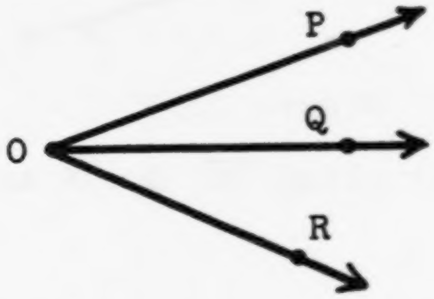
5. How close can you come to drawing the midray of an angle without using a protractor? Try it for these cases: an acute angle, a right angle, an obtuse angle.

6. Try to draw a triangle that has two right angles. If you are not able to do so, explain the failure.

#7. In this exercise, we consider what it means when three rays have the same vertex; that is to say, when one is between the other two.



(a) Look at rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, and $\overrightarrow{VC}$ in the diagram. Would you say that one of them is between the other two? If so, what would you mean?



(b) Now look at $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ in the second diagram. Would you say that one of these is between the other two?

(c) In (a) is $\overrightarrow{VA}$ a ray of $\angle BVC$? Is $\overrightarrow{VB}$ a ray of $\angle CVA$? Is $\overrightarrow{VC}$ a ray of $\angle AVB$?

(d) In (b) is $\overrightarrow{OQ}$ a ray of $\angle POR$?

(e) Formulate a definition for betweeness for rays.

#8. Draw $\angle AVB$ and an interior ray $\overrightarrow{VC}$ of this angle. We say that $\overrightarrow{VC}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Using a protractor show that $m\angle AVC + m\angle CVB = m\angle AVB$. This result is important enough to have a name. It is the Betweeness-Addition Property of Angles. State it in words. There is also a Betweeness-Addition Property of Segments. State it.

10.24 Angles and Line Reflections

Make a drawing like the one in Figure 10.47, with $\overrightarrow{VM}$ the midray of $\angle AVB$. (We have an angle of 35°. You can use any angle you like.)

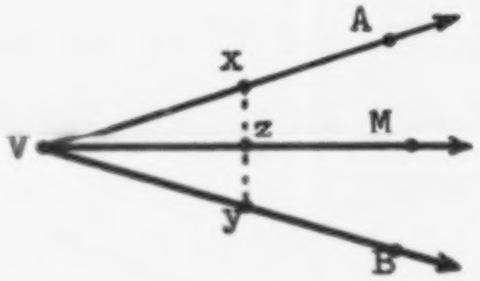


Figure 10.47

If you fold your paper along $\overrightarrow{VM}$, do $\overrightarrow{VA}$ and $\overrightarrow{VB}$ fall on each other? Then we may say:

> Each endray of an angle is the image of the other under the line reflection in the line containing the midray of the angle.

Suppose X is the point in $\overrightarrow{VA}$ such that $VX = 2$. Where would you expect to find the image of X under this line reflection ℓ? Let $X \xrightarrow{\ell} Y$. Then $VX = VY$. Moreover, the perpendicular to $\overleftrightarrow{VM}$ that contains X must also contain Y. Why? We conclude that $\overleftrightarrow{XY} \perp \overleftrightarrow{VM}$. Also if Z is the point in which $\overleftrightarrow{XY}$ intersects $\overrightarrow{VM}$, then

XZ = YZ. Why? One more result. In folding your paper, did $\angle VXY$ fall on $\angle VYX$? Then $m\angle VXY = m\angle VYX$.

Let us summarize these results. If $\overrightarrow{VM}$ is the midray of $\angle XVY$, VX = VY, and $\overleftrightarrow{XY}$ intersects $\overrightarrow{VM}$ in Z, then:

(1) under the line reflection ℓ in $\overleftrightarrow{VM}$, $V \xrightarrow{\ell} V$, $X \xrightarrow{\ell} Y$, $Z \xrightarrow{\ell} Z$. Since a line reflection is an isometry, VX = VY, XZ = YZ. Also $\overleftrightarrow{XY} \perp \overleftrightarrow{VM}$.

(2) $m\angle VXZ = m\angle VYZ$.

The second fact rates attention because it is a special case of a more general statement which we are now ready to understand. It applies to all isometries, of which line reflections are only one kind.

Under any isometry the measure of an angle is the same as the measure of its image angle.

We shall pursue this further in the next section. Meanwhile, we apply our results to a special type of triangle. If at least two sides of a triangle have the same length it is called an isosceles triangle. These two sides are called the legs of the isosceles triangle; the third side is called its base. The angles of the triangle having vertices at the ends of the base are called base angles, the third angle is called the vertex angle. Let $\triangle ABC$ be an isosceles triangle with AB = AC,

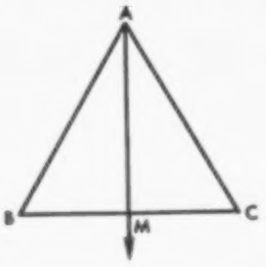


Figure 10.48

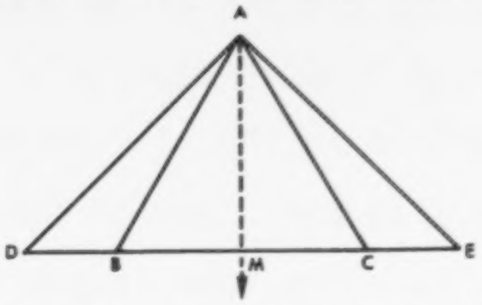
and let the midray of the vertex angle intersect the base in point M (Figure 10.48). Then under the line reflection ℓ in $\overleftrightarrow{AM}$, $A \xrightarrow{\ell} A$, $M \xrightarrow{\ell} M$, $B \xrightarrow{\ell} C$. By our previous results we conclude:

(1) The base angles of an isosceles triangle have the same measure.

(2) The midray of the vertex angle of an isosceles triangle lies in the midperpendicular of the base.

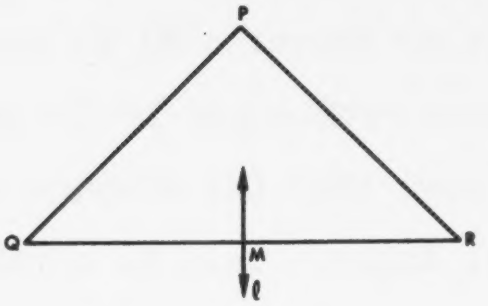
10.25 Exercises

1. Suppose D, B, C, E are on a line as shown below, and A is not. If AB = AC, show by an argument that $m\angle ABD = m\angle ACE$.

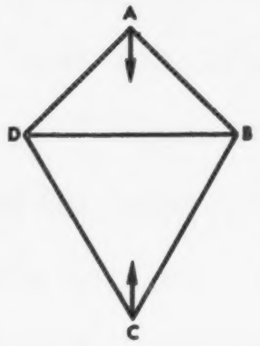


2. For the figure in Exercise 1 add the information that BD = CE. Using the line reflection ℓ in $\overleftrightarrow{AM}$, where $\overrightarrow{AM}$ is the midray of $\angle BAC$, explain why each of the following is true or false:
   (a) $\overleftrightarrow{AM}$ is the midperpendicular of $\overline{DE}$.
   (b) $E \xrightarrow{\ell} D$ and $D \xrightarrow{\ell} E$ and DM = EM.
   (c) $\overline{AD} \xrightarrow{\ell} \overline{AE}$ and AD = AE.
   (d) $\overrightarrow{AD} \xrightarrow{\ell} \overrightarrow{AE}$ and $\overrightarrow{AB} \xrightarrow{\ell} \overrightarrow{AC}$.
   (e) $m\angle DAB = m\angle EAC$.
3. Suppose PQ = PR and QM = MR as shown below. Let ℓ be the midperpendicular of $\overline{QR}$. Do you think that ℓ contains P?

Support your answer with an argument.



4. In the diagram below AD = AB and DC = CB.



(a) What kind of triangle is ABD? CBD?

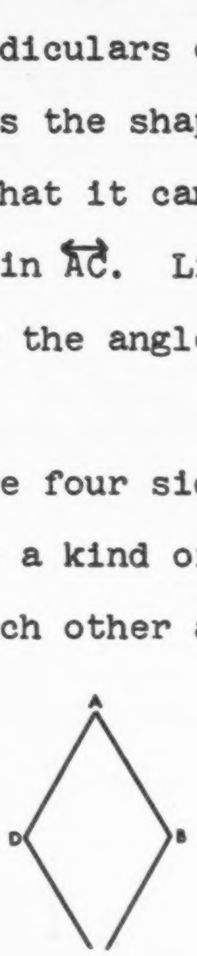
(b) How is the midray of $\angle BAD$ related to $\overline{BD}$?

How is the midray of $\angle BCD$ related to $\overline{BD}$?

(c) How many midperpendiculars of $\overline{DB}$ are there?

(d) The figure ABCD has the shape of a kite, so we call it a kite. You see that it can be mapped into itself by a line reflection in $\overleftrightarrow{AC}$. List five pairs of angles in the kite for which the angles in each pair have the same measure.

5. In the diagram below the four sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have the same length. It is a kind of "double kite." Show that its diagonals bisect each other and lie in perpendicular lines.



10.26 Angles and Point Reflections

In an exercise (10.20 Exercise 8) we noted that the image of an angle under a point reflection in its vertex is its vertical angle. It quickly follows that the measure of an angle is equal to that of its vertical angle. This is a valid conclusion. Nonetheless, let us explore the situation a little more, partly to review some basic notions and partly to illustrate a proof which resembles many that will follow.

Suppose $\angle ABC$ is a given angle (Figure 10.49). If B is the midpoint of $\overline{AA'}$ and also $\overline{CC'}$, then $\angle A'BC'$ is the image of $\angle ABC$ under a point reflection in B. We can easily locate A' and C' by using a compass with B as center. Now look at the quadrilateral ACA'C'. Its diagonals bisect each other. Then what kind of quadrilateral is ACA'C'? How does your answer lead to the conclusion that CA = C'A'?

Let us review three facts: (1) AB = A'B, (2) CB = C'B, (3) CA = C'A'. Do not these three facts show that the mapping under which $A \longrightarrow A'$, $B \longrightarrow B$, $C \longrightarrow C'$ is an isometry?

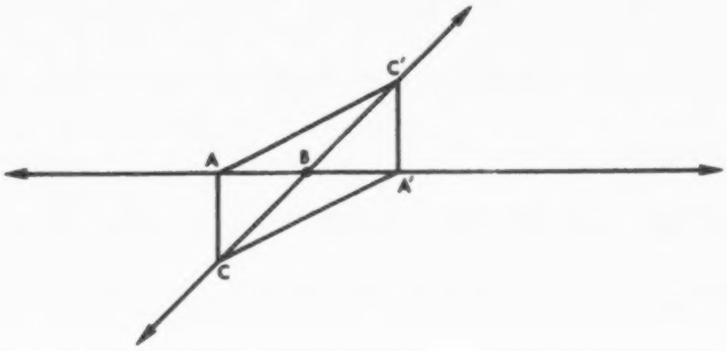


Figure 10.49

We conclude that m∠ABC = m∠A' BC' . (Remember that an isometry preserves angle measure.) In this example we reviewed the basic notion of an isometry and we have seen how to use some properties of parallelograms in a proof.

Suppose the center of a point reflection is not the vertex of an angle. In each of Figures 10.50 and 10.51, the image of ∠ABC is ∠A' B' C' under a point reflection in O, a point which is not the vertex B. Verify in each case that O is the midpoint of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. This should assure you that we do indeed have a point reflection in O.

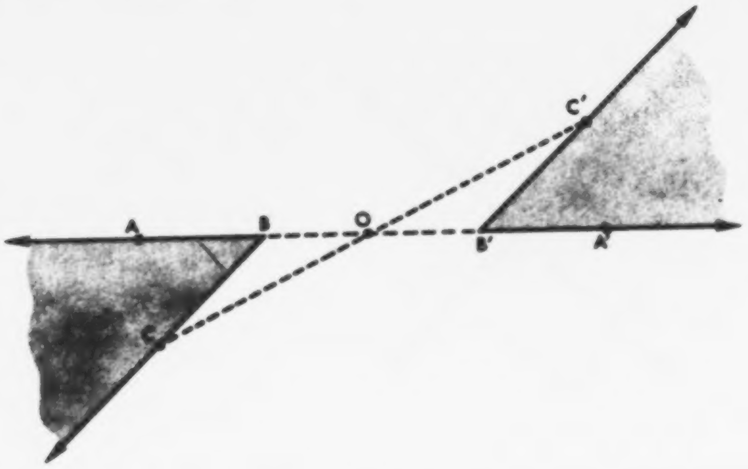


Figure 10.50

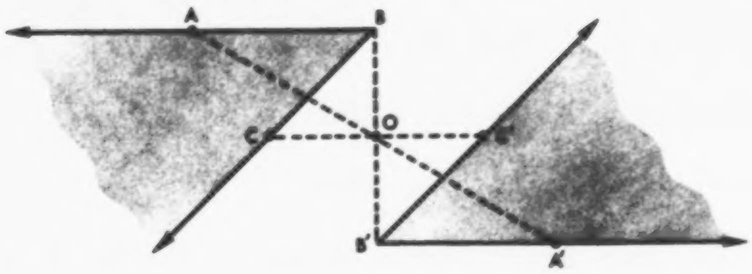


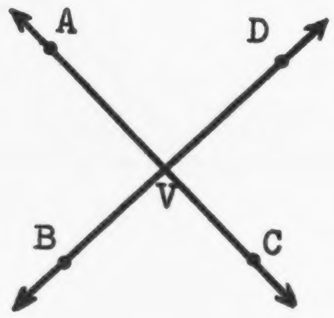
Figure 10.51

In each case the mapping of A, B, C onto A' , B' , C' respectively, can be shown to be an isometry; that is AB = A' B' , BC = B' C' and

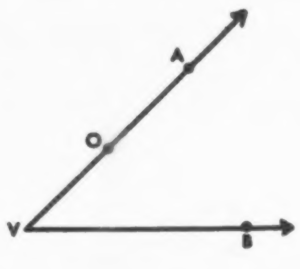
CA = C' A'. Find two parallelograms in Figure 10.50 that help to show why AC = A' C' and BC = B' C'. Try to figure out why AB = A' B'. In Figure 10.51, we can find three parallelograms that help in proving that the mapping is an isometry. Name the three parallelograms.

10.27 Exercises

1. Use a protractor to measure only one of the four angles, $\angle AVB$, $\angle BVC$, $\angle CVD$, $\angle DVA$ and then tell the measures of the other three.

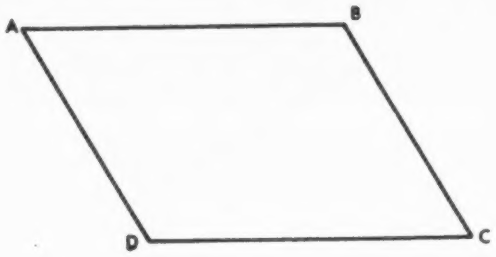


2. Draw a diagram showing the image of $\angle ABC$ under a point reflection in O for each of the following cases:
   - (a) O is a point in $\overrightarrow{BA}$, not B.
   - (b) O is a point in $\overrightarrow{BC}$, not B.
   - (c) O is an interior point of $\angle ABC$.
   - (d) O is an exterior point of $\angle ABC$.
3. Copy a figure like the one shown below. Be sure to take O as the midpoint of $\overline{VA}$. Draw the image of $\angle AVB$ under a point reflection in O.



Under this mapping what is the image of V? What is the vertex of the image angle? If B' is the image of B under the point reflection in O, show that $\overleftrightarrow{AB'} \parallel \overleftrightarrow{BV}$. The statement of this result is quite complex. We start it and you are to complete it: If the center of a point reflection of an angle is a point, but not the endpoint, of an endray of the angle, then the image of the second side ...

4. Draw an angle and its midray, and take any point, not the vertex, of its midray. Draw the image of the angle under a point reflection in this midray point. **You should** note that the angle and its image determine a quadrilateral. List some of the properties of this quadrilateral that you can find.
5. Repeat the instructions in Exercise 4 with the modification that the center of reflection in an interior point of the angle, not in the midray.
6. Suppose ABCD is a parallelogram. Is there a point reflection under which $D \longrightarrow B$, $A \longrightarrow C$? What is its center? How do your answers help to show that each angle of a parallelogram has the same measure as that of the opposite angle?



10.28 Angles and Translations

Let $\angle AVB$ be mapped by a translation such that the image of

V is V'. (See Figure 10.52.)

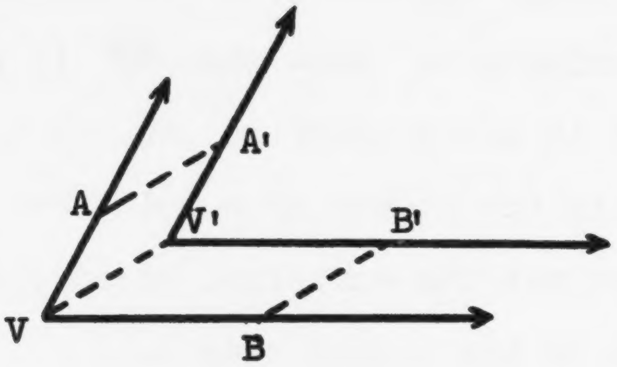
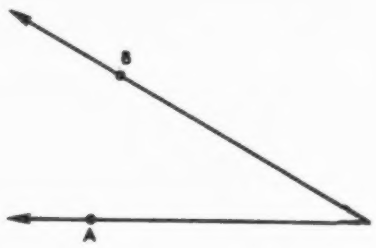


Figure 10.52

Let the images of A and B be A' and B' under this translation. Since a translation is an isometry, and we have agreed that isometries preserve angle measures, it follows that m∠A'V'B' = m∠AVB. Further results relating angles and translations are explored in the following exercises.

10.29 Exercises

#1. Copy ∠AVB and then show a translation of ∠AVB by a drawing that maps V onto A. Let the translation map A onto A' and B onto B'. Under this translation what are the images of $\overrightarrow{VA}$, $\overrightarrow{VB}$, ∠AVB? We may call the pair of angles AVB and A'AB' "F angles" because they form an F figure.

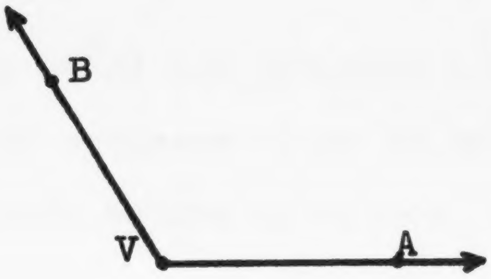


2. (a) Repeat the instructions in Exercise 1 for the translation that maps A onto V.

(b) Repeat again for the translation that maps V onto B.

3. Let T_1 be the translation that maps A onto V and T_2 the translation that maps V onto B.

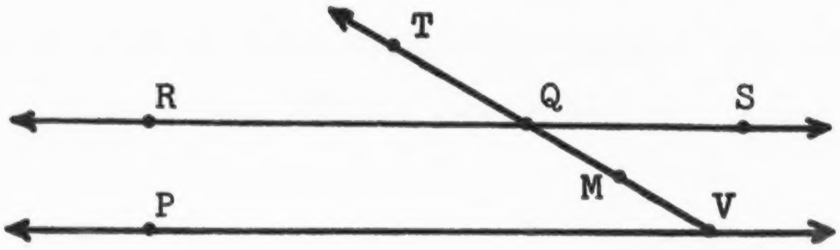
   (a) Copy the diagram below, and in it make a drawing for $T_2 \circ T_1$.



   (b) Make a drawing for $T_1 \circ T_2$ in the same diagram.

   (c) Are the images of $\angle AVB$ under both compositions the same? Are the drawings the same?

4. In the diagram below $\overleftrightarrow{RS} \parallel \overleftrightarrow{PV}$ and M is the midpoint of $\overline{QV}$.



   (a) Describe a mapping under which the image of $\angle PVQ$ is $\angle RQT$.

   (b) Describe a mapping under which the image of $\angle PVQ$ is $\angle VQS$.

   (c) Describe a mapping under which the image of $\angle RQT$ is $\angle SQV$. Is this mapping on isometry?

   (d) Describe a mapping under which the image of $\angle RQT$ is $\angle SQM$.

   (e) Under what composite mapping is $\angle SQM$ the image of $\angle PVQ$ if a translation is first in the composition?

(f) Compare the measures of $\angle PVQ$ and $\angle SQV$. We may call angles PVQ and SQV "Z angles" because they form a Z figure.

10.30 Sum of Measures of the Angles of a Triangle

No doubt you have measured the three angles of a triangle and have found the sum of their measures to be approximately 180. Let us see how isometries can be used to prove the sum is exactly 180.

Figure 10.53 shows an image for each angle of $\triangle ABC$ under different mappings.

First consider the translation that maps A onto C. This translation maps C onto R and B onto S. What are the images of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ under this translation? Do you see that this translation maps $\angle CAB$ onto $\angle RCS$?

Examine the translation that maps B onto C. Under this translation what is the image of $\overrightarrow{BA}$? of $\angle ABC$?

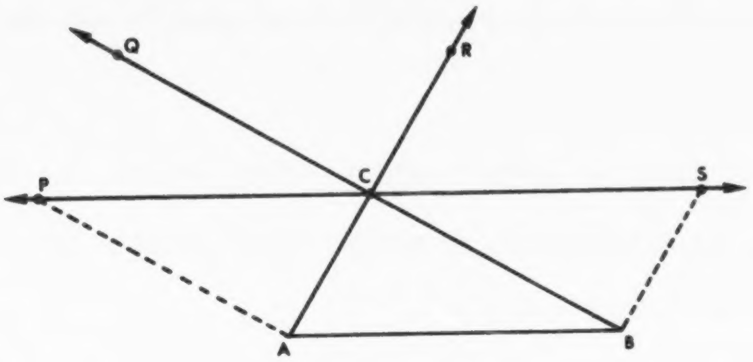


Figure 10.53

The third mapping is a point reflection in C. Under this mapping what is the image of $\angle ACB$?

As a result of these mappings, all isometries, we see:

(1) $m\angle CAB = m\angle RCS$,

(2) $m\angle ABC = m\angle PCQ$,

(3) $m\angle BCA = m\angle QCR$.

If the sum of the measures of the image angles is 180, then we can safely conclude that the sum of the measures of the angles of the triangle must also be 180.

Do you think the first sum is 180? Why? In answering this question remember that no statement was made concerning whether $\overrightarrow{CS}$ and $\overrightarrow{CP}$ were on one line. Are they? Why?

One can prove the above result by using other isometries, and you may find it interesting (in exercises) to find your own.

There are many immediate results following from the triangle measure sum. For instance we can now show: If a triangle has a right angle then the sum of the measures of the other two angles is 90. The proof can be presented in a step by step argument as follows:

(1) Let $\triangle ABC$ have a right angle at C.

(2) $m\angle A + m\angle B + m\angle C = 180$

(3) $m\angle C = 90$

(4) $m\angle A + m\angle B = 90$

We can give a valid reason for each of these statements. The reasons, numbered to let you see which reason applies to each statement, are as follows:

(1) This information is given.

(2) We have proved this already. Let us call it the Triangle Angle Sum Property.

(3) The measure of a right angle is 90.

(4) The cancellation law for addition.

Here is another immediate result with its proof:

The sum of the measures of the angles of a plane quadrilateral is 360.

Figure 10.54 will help you follow the argument.

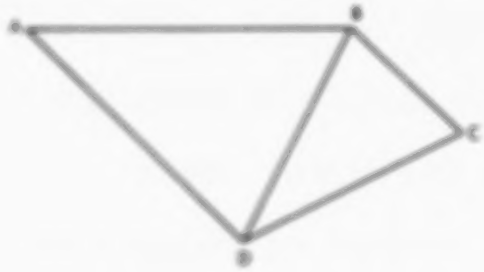


Figure 10.54

We ask you to assume that $\overrightarrow{BD}$ is an interior ray of $\angle ABC$ and $\overrightarrow{DB}$ is an interior ray of $\angle ADC$.

(1) $m\angle A + m\angle ABD + m\angle BDA = 180$

(2) $m\angle C + m\angle DBC + m\angle BDC = 180$

(3) $m\angle ABD + m\angle DBC = m\angle ABC$ or $m\angle B$

(4) $m\angle BDA + m\angle BDC = m\angle CDA$ or $m\angle D$

(5) $m\angle A + m\angle B + m\angle C + m\angle D = 360$

The reason for (1) and (2) is the Triangle Angle Sum Property. Statements (3) and (4) have the same reason, the Betweenness-Addition Property of Angles. (See Section 10.23 Exercise 8.)

The reason for statement (5) is: $180 + 180 = 360$.

In exercises you will be asked to prove many other statements which follow from the Triangle Angle Sum Property.

10.31 Exercises

1. Find the measure of the third angle of a triangle if you know the measures of the first two to be as follows:

   (a) 80 and 30 (b) 62 and 49 (c) 40 and 129

2. The measures of two angles of a triangle are the same. What is their measure if the measure of the third angle is:

   (a) 80 (b) 20 (c) 68 (d) 41

3. What is the measure of each angle of a triangle whose angles all have the same measure?

4. The measures of two angles of a triangle have the ratio 3:5. What are their measures if the third angle has a measure of:

   (a) 100 (b) 68 (c) 30

5. What is the measure of an angle of a quadrilateral if the measures of the other three angles are:

   (a) 120, 80, 62
   (b) 100, 62, 62
   (c) 168, 72, 48

6. Show that if three angles of a quadrilateral are right angles then the fourth angle must also be a right angle.

7. Let ABCD be a parallelogram. Show that $m\angle A + m\angle B = 180$ and $m\angle C + m\angle D = 180$.

8. Give an argument for each of the following statements. It need not be a step by step argument.

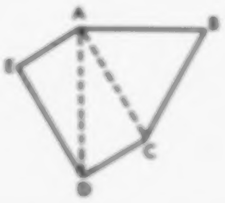
   (a) Two angles of a triangle cannot both be obtuse.
   (b) If a triangle is isosceles then its base angles are acute angles.

9. Prove each of the following. If convenient, use a step by

step argument.

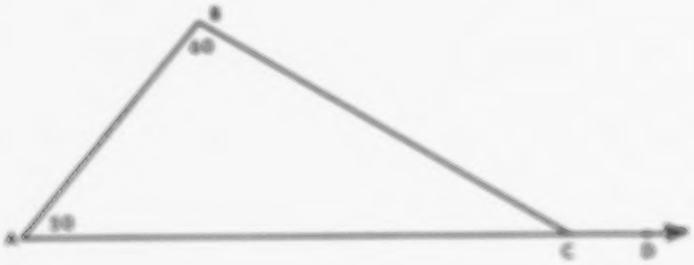
(a) If in ΔABC, $AB = BC = CA$, than $m\angle A = 60$.

(b) The figure below has 5 sides and is called a pentagon. Assume that $\overrightarrow{AD}$, $\overrightarrow{AC}$ are interior rays of $\angle EAB$, that $\overrightarrow{DA}$ is an interior ray of $\angle EDC$, and that $\overrightarrow{CA}$ is an interior ray of $\angle DCB$. Show that the sum of the measures of the angles of ABCDE is 540.



(c) Assume in (b) that the measures of all the angles in ABCDE are the same. Show that each measure is 108.

10. (a) Using the data indicated in the diagram below find $m\angle BCD$



(b) Suppose $m\angle A = 52$, $m\angle B = 65$. Again find $m\angle BCD$.

(c) Do the results in (a) and (b) suggest a relationship between $m\angle BCD$ and $m\angle A + m\angle B$?

(d) Show for all measures of $\angle A$ and of $\angle B$, that $m\angle BCD = m\angle A + m\angle B$.

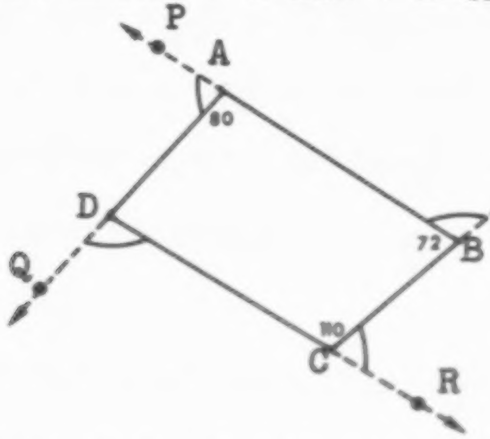
11. (a) In the diagram below, find $m\angle ADC$.

(b) Find the measures of the angles in which arcs are drawn, in the same diagram.

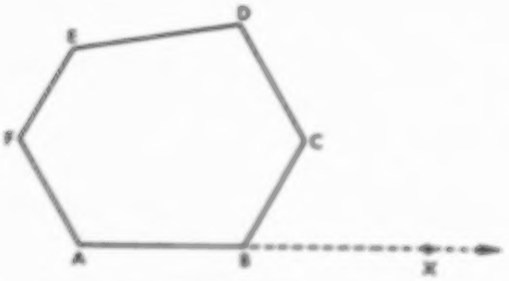
(c) Find the sum of the measures in (b).

(d) Take another set of measures for the three angles of quadrilateral ABCD and find the sum of the "arc" angles for your new measures.

(e) Do your results in (c) and (d) indicate a pattern? Complete and prove the following statement: For quadrilateral ABCD, m∠PAD + m∠RCB + m∠SBA = ?



12. A figure such as ABCDEF has six sides and is called a hexagon.



(a) Find the sum of the measures of its angles.

(b) Let X be a point in $\overrightarrow{AB}$ as shown. ∠CBX is called an exterior angle of the hexagon. Find the sum of the measures of the exterior angles of the hexagon, one taken at each vertex.

(c) If all the angles of a hexagon have the same measure, what is the measure of each angle, and what is the measure of one exterior angle?

13. Repeat Exercise 12 for a figure having 8 sides; 10 sides.

10.32 Summary

This chapter discussed segments, angles, and isometries.

1. The major items relating to segments are the following:
    - (a) The Line Separation Principle leads to subsets of lines, open halflines and rays, and then to segments.
    - (b) The distance formula: If $\underline{x_1}$ and $\underline{x_2}$ are line coordinates of A and B, then $AB = |x_1 - x_2| = |x_2 - x_1|$.
    - (c) The midpoint formula: If $\underline{x_1}$ and $\underline{x_2}$ are line coordinates of A and B, then the coordinate of the midpoint of $\overline{AB}$ is $\frac{1}{2}(x_1 + x_2)$.
    - (d) The Betweeness-Addition Property of Segments: If B is between A and C, then $AB + BC = AC$.
    - (e) The Triangle Inequality Property: The sum of the lengths of two sides of a triangle is greater than the length of the third.
2. The major items relating to angles are the following:
    - (a) The Plane Separation Principle leads to open halfplanes, halfplanes, and angles, which are intersections of halfplanes.
    - (b) The angle measure formula: When the center of a protractor is placed at the vertex of an angle, if $\underline{r_1}$ and $\underline{r_2}$ are the numbers assigned by the protractor to the two sides of the angle, the measure of the angle is $|r_1 - r_2| = |r_2 - r_1|$.
    - (c) Boxing the compass is accomplished by the repeated bisection of arcs, comparable to the bisection method

used in graduating a ruler.

(d) Angles are classified as zero, acute, right, obtuse and straight angles.

(e) The Betweeness-Addition Property of Angles: If $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, then $m\angle AVB + m\angle BVC = m\angle AVC$.

3. Isometries. A major item is: Isometries preserve angle measure.

(a) Using line reflections we can show:

(1) An angle is its own image under the line reflection in its midray. This leads to related isosceles triangle properties, and kite properties.

(2) Every point in the midperpendicular of a line segment is as far from one endpoint of the segment as from the other.

(3) The rectangular coordinate formula for the reflection in the x-axis is $(x, y) \longrightarrow (x, -y)$, for the reflection in the y-axis, $(x, y) \longrightarrow (-x, y)$.

(b) Using point reflections we can show:

(1) The measure of an angle is the same as that of its vertical angle.

(2) The measures of opposite angles of a parallelogram are the same.

(3) The angles in a "Z figure" have the same measure.

(4) The coordinate formula for the point reflection in the origin of a rectangular coordinate system is $(x, y) \longrightarrow (-x, -y)$.

(c) Under a translation we can show:

  (1) The angles in an "F figure" have the same measure.

  (2) The coordinate formula for a translation is: $(x, y) \rightarrow (x + p, y + q)$, if the origin is mapped onto (p, q).

4. Using point reflections and translations we can show why the sum of the measures of angles of a triangle is 180. This leads to a long list of immediate results.

10.33 Review Exercises

1. Let a mathematical ruler assign -2 to point A and 4 to point B.
   - (a) What is AB?
   - (b) What number does the ruler assign to the midpoint of $\overline{AB}$?
   - (c) C is a point in $\overline{AB}$. If AC + CB = AB what are the possible assignments the ruler can make to C?
   - (d) If D is between A and B and AD = 2DB what is the number assigned to D?
   - (e) What numbers may be assigned to point E if AE = 6 and E is in $\overline{AB}$?
2. In Exercise 1 replace -2, the number assigned to A, with -12 and replace 4, the number assigned to B, with -6. Answer the questions in Exercise 1 for these replacements.
3. A protractor with center at V assigns 10 to $\overrightarrow{VA}$ and 110 to $\overrightarrow{VB}$.
   - (a) What is $m\angle AVB$?
   - (b) What number does the protractor assign to the midray

of $\angle AVB$?

(c) The protractor assigns 120 to $\overrightarrow{VD}$. Is $\overrightarrow{VD}$ between $\overrightarrow{VA}$ and $\overrightarrow{VB}$?

(d) What must be true of x if x is the number assigned to a ray that is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$?

(e) Suppose $\overrightarrow{VX}$ is an interior ray of $\angle AVB$, what is $m\angle AVX + m\angle XVB$?

(f) Suppose $\overrightarrow{VY}$ is an interior ray of $\angle AVB$ such that $m\angle AVY = 2m\angle YVB$. What number does the protractor assign to $\overrightarrow{VY}$?

4. In Exercise 3 replace 10, the number assigned to $\overrightarrow{VA}$, with 122, and replace 110, the number assigned to $\overrightarrow{VB}$, with 38. Then answer the questions in Exercise 3 for these replacements.

5. Try to draw a triangle such that one of its angles is a right angle and another is an obtuse angle. Explain how you were able to or not able to make the drawing.

6. In a certain rectangular coordinate system A, B, and C have coordinates (-4, 2), (1, -3) and (6, 2) respectively.

(a) What are the coordinates of A', B', C', the images of A, B, and C, under the line reflection in the x-axis?

(b) Are A, B, C collinear? Are A', B', C' collinear?

(c) Compare AB with $A'B'$. Make the comparison without finding the numbers AB and $A'B'$ and justify your answer.

(d) Compare $m\angle ABC$ with $m\angle A'B'C'$ after measuring each angle with a protractor. Can you make the comparison without using a protractor? Justify your answer.

7. Answer the questions in Exercise 6 if A', B', and C' are the images of A, B, and C under the line reflection in the y-axis.

8. Answer the questions in Exercise 6 if A', B', and C' are the images of A, B, and C under the point reflection in the origin of the coordinate system.

9. Answer the questions in Exercise 6 if A', B', and C' are the images of A, B, and C under the point reflection in P(1, 2).

10. Answer the questions in Exercise 6 if A', B', and C' are the images of A, B, and C under the line reflection in the line perpendicular to the x-axis and containing P(1, 2).

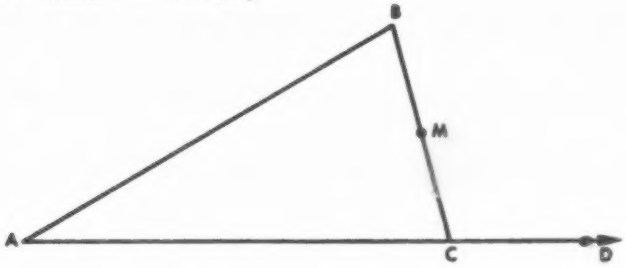
11. Consider the coordinate rule by which (x, y) is mapped onto (y, x) in a rectangular coordinate system.
    (a) Under this mapping what are the coordinates of the images of (2, 0), (0, 4), (-1, 2), (3, 3), (-5, -2), (0, 0)?
    (b) Make a graph of the points in (a) and their images.
    (c) Is this mapping a line translation, a point reflection, a translation, a point reflection, a translation, or none of these? If it is one of these, describe it, giving domain, range and the rule for its inverse mapping.
    (d) What is the composition of this mapping with itself?

12. Consider the coordinate rule in a rectangular coordinate system by which $(x, y) \longrightarrow (-y, -x)$. Answer the questions in Exercise 11 for this mapping.

13. Is the mapping with coordinate rule $(x, y) \longrightarrow (2x, 2y)$

in a rectangular coordinate system an isometry?

14. Let M be the midpoint of $\overline{BC}$ in $\triangle ABC$. Using a point reflection in M and a translation show how to prove that $m\angle A + m\angle B + m\angle C = 180$.



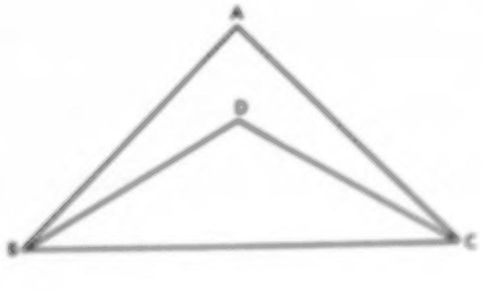
15. Find the measure of an angle of an n-sided figure, where all the angles have the same measure, and n has the value given below:

(a) $n = 6$ (c) $n = 8$ (e) $n = 20$

(b) $n = 3$ (d) $n = 12$

16. Find the measure of an exterior angle of each n-sided figure in Exercise 15.

17. In the figure below $AB = AC$, and $DB = DC$. Using a line reflection, prove $m\angle DAB = m\angle DAC$.



CHAPTER 11

ELEMENTARY NUMBER THEORY

11.1 (N, +) and (N, ·)

Over the centuries many discoveries have been made concerning properties that various sets of numbers possess. In this chapter we shall concentrate on seeking out properties of certain subsets of the whole numbers. In particular we shall examine the set of natural numbers. (By the natural numbers, N, we mean the whole numbers with zero deleted.)

$$N = \{1, 2, 3, . . . \}$$

We shall begin by stating certain basic assumptions concerning the natural numbers. Such assumptions, that is statements which we agree to accept as true without proof, are called _axioms_. We shall use these axioms to prove other statements which we call _theorems_. In fact, number theory provides us with a large source of simple and important theorems from which we can begin to learn some of the basic ideas dealing with "proof."

Before stating the first axiom let us recall a problem considered in Chapter 2 (Section 2.4, Exercise 12): "Is addition an operation on the set of odd whole numbers?" It is easy to find an example which indicates the answer to this question is "No." Both 3 and 5 are odd whole numbers but their sum, 8, is not an odd whole number. Because the set of odd whole numbers is a subset of W we see that addition is not an operation on every subset of W. Thus any statement which asserts that addition is an operation on a subset of W is a non-trivial statement. Our first axiom (A1) states that addition is an operation on N.

A1. (N, +) is an operational system.

Because 3∈N and 5∈N we can conclude, by A1, that 3 + 5 = 8 ∈ N.

In general A1 states that to each ordered pair of natural numbers addition assigns exactly one natural number called their sum.

An obvious question to consider next is the following: "Is multiplication an operation on N?" Our second axiom provides the answer to this question.

A2. (N, ·) is an operational system.

Since 3∈N and 5∈N we can conclude by A2 that 3·5 = 15∈N. In general, A2 states that to each ordered pair of natural numbers multiplication assigns exactly one natural number called their product.

For example:

$$(3,5) \xrightarrow{\cdot} 15$$

We frequently express the above by the mathematical sentences

$$3 \cdot 5 = 15 \qquad \text{or} \qquad 3 \times 5 = 15.$$

Let us review some of the language used in discussing the operational system (N, ·). In the sentence above, 3 is said to be a factor of 15. Also, 5 is said to be a factor of 15.

Definition 1: We say that for a and b in N, a is a factor of b if and only if there is some natural number c such that $a \cdot c = b$.

Thus 3 is a factor of 15 because there is a natural number, 5, such that $3\cdot 5 = 15$. 4 is not a factor of 15 because there is no natural number c such that $4\cdot c = 15$. 5 is a factor of 15 because $5\cdot 3 = 15$.

Recall that in Chapter 2 you were introduced to the idea of *multiple*. For the mathematical sentence

$$3\cdot 5 = 15$$

we say that 15 is a multiple of 3 and also that 15 is a multiple of 5.

Definition 2: For *a* and *b* in N, *b* is a multiple of *a* if and only if *a* is a factor of *b*.

Thus for the mathematical sentence

$$4\cdot 9 = 36$$

we can make the following statements:

4 is a factor of 36
9 is a factor of 36
36 is the product of the factors
4 and 9
36 is a multiple of 4
36 is a multiple of 9

In Chapter 6 we made frequent use of the binary relation "divides" on various sets of numbers. In this chapter we again make use of this relation. In particular, if 4 is a factor of 36 we say that 4 *divides* 36 and we write

$$4 \mid 36$$

Definition 3: We say that for *a* and *b* in N, *a* divides *b* if and only if *a* is a factor of *b*. We denote "a divides b" by "a | b."

For the sentence

$$3 \cdot 4 = 12$$

We can make the following statements:

3 is a factor of 12
3 divides 12
$3 \mid 12$
4 is a factor of 12
$4 \mid 12$
12 is a multiple of 4, etc.

Since 5 is not a factor of 12 we can say that 5 does not divide 12 (written $5 \nmid 12$).

Because $1 \cdot n = n$ where n is any natural number we see that 1 is a factor of every natural number. Also, every natural number is a multiple of 1.

Question: Can we say that $1 \mid n$ for all n in N? Explain.

You are familiar with the idea that every natural number has many names. A number such as 12 can be renamed in many ways:

$10 + 2$ $\qquad$ $3 \cdot 4$
$1 \cdot 12$ $\qquad$ $6 \cdot 2$

We shall use the words product expression to talk about names such as "$1 \cdot 12$" and "$3 \cdot 4$" that involve multiplication. We say that "$1 \cdot 12$" and "$3 \cdot 4$" are product expressions of 12. It is possible to have product expressions for 12 with more than two factors such as:

$1 \cdot 2 \cdot 6$ $\qquad$ $2 \cdot 2 \cdot 3$
$1 \cdot 3 \cdot 4$ $\qquad$ $1 \cdot 2 \cdot 2 \cdot 3$

We see that we can use any of several different product expressions to represent the number 12.

Question: How many product expressions of 12 are there which contain exactly two factors?

Question: Is $59 \cdot 509$ a product expression of 30031? (The number 30031 will be mentioned later in this chapter in connection with an important theorem)

11.2 Exercises

1. Explain why the following are, or are not, true:
    (a) $(2 + 3) \in N$
    (b) $(2 \cdot 3) \in N$
    (c) If $a \in W$ and $b \in W$, then $(a + b) \in N$
    (d) If $x \in N$ and $y \in N$, then $(x + y) \in N$
    (e) If $p \in N$ and $q \in W$, then $(p \cdot q) \in N$
    (f) The product of two natural numbers is a natural number.
2. Complete the following sentences:
    (a) If $\underline{a}$ is a factor of $\underline{b}$, then $\underline{b}$ is a ________ of $\underline{a}$.
    (b) If $x \cdot y = z$, then ________ is a factor of ________.
    (c) If $p \cdot q = r$, then ________ is a multiple of ________.
    (d) If $5 \mid 100$, then 5 is a ________ of 100.
    (e) If $7 \cdot 8 = 56$, then 56 is called the ________ of ____ and __________.
    (f) If $9 \cdot 7 = 63$, then "$9 \cdot 7$" is called a ________ of 63.
3. Determine if the following are or are not true. Explain your answers.

(a) 3 is a factor of 18

(b) 7 is a factor of 17

(c) 3 is a factor of 10101

(d) 12 is a factor of 96

(e) 30 is a factor of 510

(f) 1 is a factor of 3

(g) 8 is a factor of 8

(h) 65 is a multiple of 13

(i) 91 is a multiple of 17

(j) 5402 is a multiple of 11

(k) 10 is a factor of 1000 because $10 \cdot 100 = 1000$

(l) 16 is a factor of 8 because $8 \cdot 2 = 16$

4. Determine if the following are or are not true. Explain your answer.

(a) $3 \mid 39$

(b) $17 \mid 91$

(c) $8 \mid 4$

(d) $1 \mid 4$

(e) $13 \mid 65$

(f) $3 \mid 6$, $3 \mid 12$ and $3 \mid 18$

(g) $2 \mid n$ where _n_ is any even natural number

(h) $n \mid n$ where _n_ is any natural number

(i) $n \mid n^2 + 3n$ for all _n_ in N

5. For the following numbers determine all product expressions which contain exactly two factors.

(a) 6 (c) 1

(b) 7 (d) 12

(e) 13	(h) 35
(f) 2	(i) 36
(g) 3	(j) 37

11.3 Divisibility

In this section we shall consider how some sentences dealing with natural numbers can be established as theorems. An example of such a sentence is the following:

If a is an even natural number and b is an even natural number then a + b is an even natural number. Our goal is to prove that a + b must be an even natural number whenever a and b are even natural numbers. In order to prove this some additional axioms for (N, +, ·) are needed. Rather that just stating those axioms needed to prove the above sentence, we now record a number of additional axioms for (N, +, ·) which may be used to prove many other theorems.

A3. For all a and b in N, $a + b = b + a$ and $a \cdot b = b \cdot a$.

A4. For all a, b, and c in N,
$a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

A5. For all a, b, and c in N,
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

A6. For all a in N, $a \cdot 1 = 1 \cdot a = a$.

Question: What familiar names do we give to the axioms A3 - A6?

Besides these properties of natural numbers, we will make frequent use of a general logical principle called the Replacement Assumption.

> The mathematical meaning of an expression is not changed if in this expression one name of an object is replaced by another name for the same object.

As an illustration, consider the use of the cancellation property in solving the equation $7.2 + x = 46$. Another name for 46 is $(7.2 + 38.8)$. Therefore, using the Replacement Assumption, we can write

$$7.2 + x = 7.2 + 38.8$$

and conclude that $x = 38.8$.

There are two specific ways in which the replacement assumption will be used in establishing proofs of sentences about the natural numbers. These are contained in the following theorem.

Theorem A. If a, b, c, and d are natural numbers such that $a = b$ and $c = d$, then

1. $a + c = b + d$
2. $a \cdot c = b \cdot d$

Proof:

1. Clearly, $a + c = a + c$. Since $c = d$ means that "c" and "d" are two names for the same object, we can replace any "c" by "d" without changing the mathematical meaning of the expression involved. Using this replacement we have $a + c = a + d$. Similarly, since $a = b$ means that "a" and "b" are names for the same object, we can replace any "a" by "b" without changing the mathematical meaning of the expression involved. Therefore, $a + c = b + d$.

Note that the two replacements were made for the "a""and "c" to the right of the "=" in $a + c = a + c$.

2. To show that $a \cdot c = b \cdot d$ we proceed in a similar manner. Certainly $a \cdot c = a \cdot c$. Replacing "c" with "d" and "a" with "b" to the right of the "=" we obtain $a \cdot c = b \cdot d$.

Let us now consider how we can prove the sentence about even natural numbers with which we began this section. Before beginning the proof we note that a natural number n is defined to be even if and only if $2 \mid n$. Our proof proceeds as follows.

Since a is an even natural number, we know that $2 \mid a$ or that 2 is a factor of a. By Definition 1 this means that there is a natural number x such that $a = 2 \cdot x$. Similarly, since b is an even natural number, $2 \mid b$ and there is a natural number y such that $b = 2 \cdot y$. Then, by the first part of the Theorem A, $a + b = 2 \cdot x + 2 \cdot y$. But $2 \cdot x + 2 \cdot y = 2 \cdot (x + y)$ by the Distributive Property, A5. Hence, we may use the replacement assumption to obtain $a + b = 2 \cdot (x + y)$. Since $x \in N$ and $y \in N$ then, by A1, $(x + y) \in N$. We see that according to Definition 1 this means that $2 \mid (a + b)$. Hence $a + b$ is an even natural number and the proof is complete.

We can also express the above in the following manner using "parallel columns." That is, statements used in the "proof" appear in the left column and justifications of these statements appear in the right column.

Theorem: If $2 \mid a$ and $2 \mid b$, then $2 \mid (a + b)$, where a and b are natural numbers.

Proof:

1. $2 \mid a$ and $2 \mid b$	1. Given
2. $a = 2x$ and $b = 2y$ where $x, y \in N$	2. Definition 1
3. $a + b = 2x + 2y$	3. Theorem A
4. $2x + 2y = 2(x + y)$	4. A5
5. $a + b = 2(x + y)$	5. Replacement Assumption
6. $(x + y) \in N$	6. A1
7. $2 \mid (a + b)$	7. Definitions 3 and 1

We call the above "a proof" of the theorem

$$\text{If } 2 \mid a \text{ and } 2 \mid b, \text{ then } 2 \mid (a + b). \qquad (1)$$

We mean that we have shown that the conditional sentence (1) (i.e., a sentence of the "if p, then q" type) is true for all values of the variables a and b. It is possible to generalize sentence (1) to obtain

$$\text{If } c \mid a \text{ and } c \mid b, \text{ then } c \mid (a + b)$$
$$\text{where } a, b, c \in N \qquad (2)$$

In order to give a proof of (2) one must show that it is true for all natural numbers a, b, and c. (This will be asked for in an exercise.)

Question: Would sentence (2) be proven as a theorem if we proved it true for $c = 3$?

We have settled the question concerning the sum of any two even natural numbers. But what can be said concerning the product of two such numbers? A little experimentation (e.g., $2 \cdot 4 = 8$, $6 \cdot 8 = 48$, etc.) suggests the following theorem:

Theorem: If $2 \mid a$ and $2 \mid b$, then $2 \mid (a \cdot b)$, where a and b are natural numbers.

A proof follows, just like the one for the last theorem. Cover up the reasons for the proof and see if you can supply them yourself. Look if you feel you have to or if you want to check your reasons.

Proof:

1.	$2 \mid a$ and $2 \mid b$	1.	Assumption (or given)
2.	$a = 2x$ and $b = 2y$ where $x, y \in N$	2.	Definitions 3 and 1
3.	$a \cdot b = (2x) \cdot (2y)$	3.	Theorem A
4.	$(2x) \cdot (2y) = 2[x \cdot (2y)]$	4.	A4
5.	$a \cdot b = 2[x \cdot (2y)]$	5.	Replacement [Statement 3 & 4]
6.	$[x \cdot (2y)] \in N$	6.	Statement 2 and A2
7.	$2 \mid (a \cdot b)$	7.	Definition 1

Sometimes we use a single letter symbol, such as "p" or "q" to represent a whole phrase or sentence. Thus we may write:

"Two divides a and two divides b" in the shorter form

"$2 \mid a$ and $2 \mid b$"

or replace this expression by the symbol "p" where

"p" means "$2 \mid a$ and $2 \mid b$."

Similarly we could use "q" to mean "$2 \mid a \cdot b$" or "two divides the product of a by b." Thus we can represent the preceding theorem by:

If p, then q.

We refer to "p" as the "hypothesis" and to "q" as the "conclusion."

In order to prove (3) we assume that p was true. That is, we assumed that the hypothesis "$2 \mid a$ and $2 \mid b$" was true. Then, using our axioms and definitions, we proceeded to establish that the conclusion "$2 \mid (a \cdot b)$" was true.

The direct method of proof is one of several accepted methods of establishing mathematical sentences as theorems. Often the direct method is not the simplest way to prove a sentence true. Another method of proof, called the indirect method, is useful in many instances. To illustrate the method we shall apply it to proving the following theorem.

Theorem: If a and b are natural numbers, and $a \cdot b$ is an odd natural number, then a and b are both odd natural numbers. (4)

(An odd natural number is any natural number that is not even.)

Proof:

As before, we begin by assuming that $a \cdot b$ is an odd natural number. But rather than using this fact directly we now ask whether it is possible for one of a or b to be even? To answer this question we consider first the possibility that a is even. If a is even, $a = 2 \cdot x$, $x \in N$. Then, $a \cdot b = (2 \cdot x) \cdot b = 2 \cdot (x \cdot b)$ which means that $a \cdot b$ is

> even. But a·b is odd. Hence, a cannot be even. Hence a must be odd. In a similar fashion we see that b cannot be even. Therefore, both a and b must be odd if a·b is odd, and our proof is complete.

In order to prove (4) we assumed that the hypothesis was true, that a·b was odd. Then we considered the possibility that the conclusion might be false, that is, that a was even or b was even. In either case this could not be true because it meant that a·b was even. We thus reasoned that the conclusion must be true.

The above proof concerning odd natural numbers made use of the definition of odd natural numbers as natural numbers which are not even. It is possible to give a more satisfactory definition of odd numbers. For this definition we will need to review some ideas studied in your earlier work with arithmetic. In particular recall that when you were asked to divide a natural number by another natural number you frequently expressed the answer in terms of a quotient and a remainder. Consider the following two displays of work done to divide 15 by 2:

$$\begin{array}{rr} & 6 \\ 2 & \overline{)15} \\ & \underline{12} \\ & 3 \end{array} \qquad\qquad \begin{array}{rr} & 7 \\ 2 & \overline{)15} \\ & \underline{14} \\ & 1 \end{array}$$

In both displays we obtain a quotient and a remainder. On the left we have a quotient 6 and a remainder 3, whereas on the right we have a quotient 7 and a remainder 1. For the display on the left we have:

$$15 = (6 \cdot 2) + 3$$

For the display on the right we have:

$$15 = (7 \cdot 2) + 1$$

In a sense we have two "answers" for our division problem involving a quotient and a remainder. We resolve this situation of not having a unique answer by saying that we will accept only that result in which the remainder is a whole number and is less than the divisor. Then the display on the left is unacceptable because the remainder 3 is not less than the divisor 2. The question of whether we can always find exactly one quotient and exactly one remainder when a whole number is divided by a natural number is answered by the following axiom which is known as the Division Algorithm.

A7. Let a be a whole number and b be a natural number. Then there is exactly one pair of whole numbers q and r such that

$$a = (q \cdot b) + r \text{ with } 0 \leq r < b.$$

Example 1: Let $a = 39$ and $b = 9$. Then the division algorithm (A7) guarantees that whole numbers q and r exist such that

$$39 = (q \cdot 9) + r \text{ with } 0 \leq r < 9.$$

In fact if we let $q = 4$ and $r = 3$ we have

$$39 = (4 \cdot 9) + 3 \text{ with } 0 \leq 3 < 9.$$

Moreover, the division algorithm guarantees that $q = 4$ and $r = 3$ are the only whole numbers which satisfy

$$39 = (q \cdot 9) + r \text{ with } 0 \leq r < 9.$$

Example 2: Consider a case where *a* is less than *b*.
If $a = 8$ and $b = 17$, then

$$8 = (0 \cdot 17) + 8$$

where the quotient is 0 and the remainder is 8. Note that the remainder is a whole number and is less than the divisor. That is $0 \leq 8 < 17$.

Example 3: If a whole number is divided by 2, the division algorithm guarantees that there exists exactly one pair of whole numbers *q* and *r* such that

$$a = (q \cdot 2) + r \text{ where } 0 \leq r < 2.$$

It is clear that the only possible values of *r* are 0 and 1. Thus we have

either $\quad a = (q \cdot 2) + 0 \quad$ (1)

or $\quad a = (q \cdot 2) + 1 \quad$ (2)

We can use the above to give us the following:

Definition 4: (a) *n* is an even whole number if and only if *n* can be expressed as $n = (q \cdot 2) + 0$, where *q* is some whole number.

(b) *n* is an odd whole number if and only if *n* can be expressed as $n = (q \cdot 2) + 1$, where *q* is some whole number.

In other words, an even whole number is twice some whole number, while an odd whole number is one more than some even whole number.

It is easy to establish the following:

Let $E = \{x \mid x \text{ is an even natural number}\}$

and $O = \{y \mid y \text{ is an odd natural number}\}$.

Theorem: (a) If $a \in E$ and $b \in O$, then $(a + b) \in O$.

(b) If $a \in O$ and $b \in O$, then $(a + b) \in E$.

(c) If $a \in E$ and $b \in O$, then $(a \cdot b) \in E$.

(d) If $a \in O$ and $b \in O$, then $(a \cdot b) \in O$.

The proof of the above will be called for in the exercises.

We conclude this discussion of odd and even natural numbers with a theorem whose proof makes use of Definition 4 and the above theorem. It also illustrates a method of proof sometimes called proof by cases.

Theorem: If n and $n + 1$ are natural numbers, then $n(n + 1)$ is an even natural number.

Proof: $n(n + 1) = n^2 + n$ (by A5 and by definition of n^2).

(1) If n is even, then n^2 is even. If n and n^2 are even, then $n^2 + n$, as the sum of two even natural numbers, is even.

(2) If n is odd, n^2 is odd, and if n and n^2 are odd, then $n^2 + n$, as the sum of two odd natural numbers, is even.

Hence, in either case (1) or case (2), $n^2 + n$ is even. Since $n(n + 1) = n^2 + n$, $n(n + 1)$ is even.

Question: Why does the above proof consider only two cases?

11.4 Exercises

1. Complete the following:

(a) $a = (q \cdot b) + r$, $0 \le r < b$, is called the ____________.

(b) $(x + 1) \cdot y = x \cdot y + y$ follows from ____________.

(c) $7 \cdot 1 = 7$ follows from __________________.

(d) If $x = y$ and $p = q$, then $x + p = y + q$ follows from ______________________.

(e) 7 is an odd natural number because __________________.

(f) If a is an odd natural number, then $a =$ ______________.

(g) If q is false implies p is false, then ______________.

(h) If $k \in N$ and $i \in N$, then $(k,i) \in N$ follows from _____.

2. Find all possible pairs of whole numbers q and r such that $13 = (3 \cdot q) + r$. Which of these pairs are the quotient and remainder of the division algorithm? For which case (s) does r satisfy $0 \leq r < 3$?

3. (a) Prove if $3 \mid a$ and $3 \mid b$, then $3 \mid (a + b)$, where $a, b, \in N$.

   (b) Prove if $c \mid a$ and $c \mid b$, then $c \mid (a + b)$, where $a, b, c \in N$.

4. Prove if $a \mid b$ and $b \mid c$, then $a \mid c$ where $a, b, c \in N$.

5. Prove if $a \mid b$, then $a \mid (bc)$, where $a, b, c \in N$.

6. Let E and O represent respectively the set of even natural numbers and the set of odd natural numbers.

   Prove: (a) If $a \in E$ and $b \in O$, then $(a + b) \in O$.

   (b) If $a \in O$ and $b \in O$. then $(a + b) \in E$.

   (c) If $a \in E$ and $b \in O$, then $(a \cdot b) \in E$.

   (d) If $a \in O$ and $b \in O$, then $(a \cdot b) \in O$.

7. Find three odd numbers totaling 30, or else prove that no such odd numbers exist.

8. Examine each of the statements (a), (b), and (c). If the statement is false then exhibit a counterexample. If the

statement is true then list all the assumptions that you need in order to complete a proof of the statement.

(a) If $a \mid b$, then $a \mid (b + c)$.

(b) If $a \mid b$, then $a \mid (bc)$.

(c) If $a \mid (b + c)$ and $a \mid b$, then $a \mid c$.

9. In this problem we consider some tests that may be applied to divisibility questions involving base ten. These tests will generally fail when numbers are represented with numerals in bases different from ten.

Assume the following is true for natural numbers a, b_1, b_2, ..., b_m:

If $a \mid b_1$, $a \mid b_2$, ..., $a \mid b_{m-1}$ and if $a \mid (b_1 + b_2 + \dots b_{m-1} + b_m)$ then $a \mid b_m$.

Also note that any natural number N can be written in the form $N = a_n 10^n + a_{n-1}\ 10^{n-1} + \dots + a_2 10^2 + a_1 \cdot 10 + a_0$; where a_0, a_1, ..., a_n and n are natural numbers.

(a) Prove that a natural number is divisible by 2 if and only if the last digit of its (base ten) numeral is even.

(b) Note $3 \mid (10-1)$, $3 \mid (10^2-1)$, $3 \mid (10^3-1)$, etc. Assume $3 \mid (10^k-1)$ where k is any natural number. Prove a natural number is divisible by 3 if and only if the sum of the digits of its (base ten) numeral is divisible by 3. [Hint: $10^k = 10^k - 1 + 1$.]

(c) Discover a decimal numeral test which indicates when a number is divisible by:

(1) 4 (2) 5

(3) 6	(5) 9
(4) 8	(6) 10

(d) Prove any of the results you have discovered in (c).

11.5 Primes and Composites

It is obvious that the natural number 8 has more factors than the natural number 7. The set of factors of 8 is {1, 2, 4, 8} whereas the set of factors of 7 is {1, 7}. It is not hard to find other natural numbers like 7 which have exactly two distinct numbers in their factor set. For example, 11 is such a number since the set of factors of 11 is {1, 11}. 2 is another natural number with precisely two numbers in its set of factors. Such numbers as 2, 7, and 11 are called prime numbers. In general, we have the following:

Definition 5: A natural number is said to be a prime number if the number has two and only two distinct factors -- namely, 1 and the number itself.

Example 1: 3 is a prime number since the only factors of 3 are 1 and 3.

Example 2: 31 is a prime number since the only factors of 31 are 1 and 31.

Example 3: 91 is not a prime number because 91 = 7 x 13. That is, 91 has factors other than 1 and 91.

Example 4: 1 is not a prime number. What in the definition of prime number determines that 1 is not a prime?

We see from Example 4 that the smallest prime number is 2. Are the multiples of 2 which are greater than 2 prime numbers? We know that 4 is a multiple of 2. But 4 cannot be a prime number because it has a factor other than 1 and itself, namely 2. Similarly, 6, being a multiple of 2, has a factor, 2, other than 1 and 6 and thus cannot be a prime number. In general, no multiple of 2 except 2 can be a prime number. Why?

What about multiples of the prime number 3? Can they ever be prime numbers? If we examine any multiple of 3 greater than 3, say 9 or 21 or 3000, we see that every such multiple has a factor other than 1 and itself, namely 3. In short, there are many natural numbers which are not prime. We call numbers of this type composite numbers. A composite number always has numbers in its factor set besides 1 and the number itself. The factor set for the composite number 9 is {1, 3, 9}.

Definition 6: A natural number is a composite number, if it is not 1, and it is not a prime number.

Example 1: The natural number 51 is a composite number. Clearly 51 is not 1. Also, 51 is not a prime number because it has the factors 3 and 17. We note that the factor set of 51, {1, 3, 17, 51}, has more than two elements.

Example 2: All multiples of 5, except 5, are composite. That is {10, 15, 20, 25, 30 ...} consists of composite numbers. Why?

Example 3: The natural numbers 90, 91, 92, 93, 94, 95, 96, 98, and 99 are all composite. Check that 97 is a prime number.

From the remarks and examples above it can be seen that we now have a partition (see Section 8.15) of the set of natural numbers into three disjoint subsets. These subsets are the following:

(i) the set consisting of 1 alone, that is {1}.

(ii) the set of prime natural numbers.

(iii) the set of composite natural numbers.

11.6 Exercises

1. Complete the following sentences:

   (a) If a natural number is a prime number, then its factors are ______________________.

   (b) If a natural number is not a prime number, then it is ______________________.

   (c) If a natural number is a prime number, then it has ____________________ elements in its set of factors.

   (d) If a natural number is not a prime number, then it has ____________________ elements in its factor set.

2. List the set of factors for the following natural numbers:

   (a) 10
   (b) 13
   (c) 12
   (d) 24
   (e) 34
   (f) 35
   (g) 36
   (h) 37

3. Determine which of the numbers given in Exercise 2 are

   (a) prime;

(b) composite;

(c) both prime and composite.

4. What can be said about every multiple of a prime number which is greater than that prime number?
5. (a) What is the greatest prime number less than 50?

   (b) What is the smallest composite number?
6. What can be said about the product of two prime numbers?
7. (a) List the set of all even prime numbers.

   (b) List the set of all odd prime numbers less than 20.
8. Re-examine the definition of composite number. Try to formulate a different definition which makes use of the term "factor" or "factor set"?
9. Find three composite numbers, each of which has

   (a) 3 numbers in its factor set;

   (b) 4 numbers in its factor set.

11.7 Complete Factorization

As you continue your study of the set of natural numbers and their properties you will frequently have to examine the factors that make up the product expressions of a natural number. What can we say about the factors that make up the product expressions of prime numbers? We have seen that

$$2 = 1 \cdot 2$$

$$3 = 1 \cdot 3$$

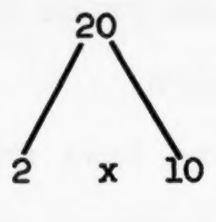
$$5 = 1 \cdot 5, \text{ etc.}$$

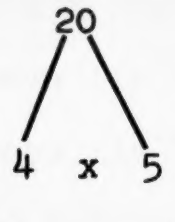
By the definition of prime numbers the only factors a prime $\underline{p}$ has are 1 and $\underline{p}$. However, we find that every composite number can be renamed as a product expression other than 1 times the number. For example, 20 can be renamed using either of the following product expressions:

$$2 \cdot 10 \qquad (1)$$

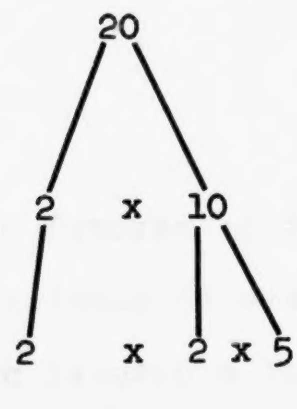
$$4 \cdot 5 \qquad (2)$$

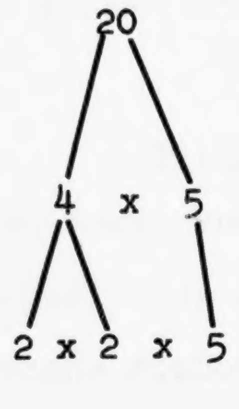
These product expressions of 20 can be shown in another way:





On the left we have a tree diagram to represent (1) and on the right a tree diagram to represent (2). It is possible to continue each of the above diagrams by completing another row to indicate product expressions of 20 as follows:



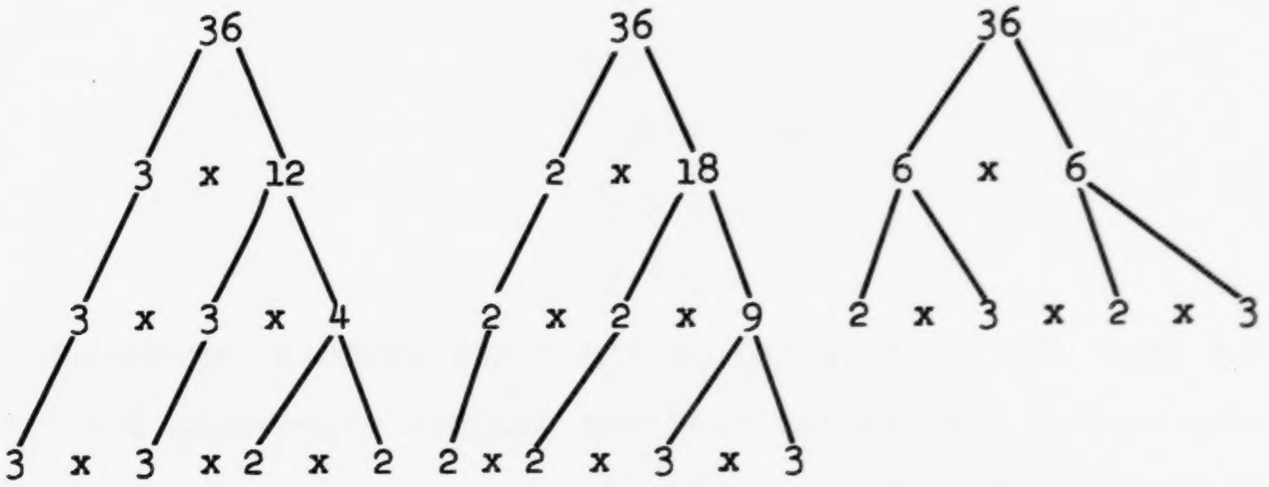


We see that every number named in the last row of both diagrams is a prime number. (We shall refer to such tree diagrams as factor trees.) Moreover, the last row in both factor trees contain exactly the same prime numbers. Thus, starting with either of the product expressions (1) or (2) of 20 we obtain exactly the same prime product expression of 20. In this case we see that 20 has a product expression such that each factor that makes up the product expression is a prime number. We shall describe this situation by saying that 20 is expressed as a product of prime factors.

Our attention is directed to the following questions:

Can every composite number be expressed as a product of prime factors? In other words, does there exist a product expression for each composite number in which each factor is a prime number? Furthermore, is there only one such product expression?

The following factor trees for 36 suggest that the answer to the above questions should be "Yes."



We note again that the last row in each of the above factor trees is a product expression for 36 in which each factor is a prime number. Moreover, the same set of factors appear in each product expression. Note that the order of the factors in each of the last rows of the factor trees is different. Is this change in the order of the factors a significant change? The answer is "No." Because multiplication is commutative and associative in (N, ·), the fact that they are arranged in different order is immaterial. Thus, using exponents, we can express the last row in each of the above tree diagrams as:

$$2^2 \cdot 3^2$$

When a composite number is expressed as a product of prime factors, we refer to this as a complete factorization of the given number.

The following are examples of complete factorizations:

$$\begin{aligned} 72 &= 2 \cdot 36 \\ &= 2 \cdot 2 \cdot 18 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \\ 182 &= 2 \cdot 91 \\ &= 2 \cdot 7 \cdot 13 \\ 150 &= 2 \cdot 75 \\ &= 2 \cdot 3 \cdot 25 \\ &= 2 \cdot 3 \cdot 5 \cdot 5 \end{aligned}$$

Notice that when each factor in the final product expression is a prime number then we say that the product expression for complete factorization has been found.

One important question that can be asked is the following: If a composite number has a complete factorization, could it have a second complete factorization involving different prime numbers? All the examples considered above seen to indicate that there is only one complete factorization for a given composite number.

For example consider:

$$150 = 2 \cdot 3 \cdot 5 \cdot 5$$

If you experiment with other possible prime factors, such as 7, 11, 13, etc., you will find that the above is the only complete factorization of 150.

The above examples illustrate one of the most important and fundamental properties of the set of natural numbers. The property is called Unique Factorization of the Natural Numbers:

> Every natural number greater than 1 is either a prime or can be expressed as a product of primes in one and only one way, except for the order in which the factors occur in the product.

We shall see how this property can be used to solve, in a new way, a problem that you met earlier in this course.

There was an exercise in Chapter 2 (see Section 2.2, Exercise 12) in which you were to find the greatest common divisor of 24 and 16. It turns out that finding the greatest common divisor of two natural numbers is equivalent to finding the greatest common factor of the two numbers. We can redefine a greatest common divisor of two natural numbers using the terminology of this chapter.

Definition 7: The greatest common divisor (abbreviated g.c.d.) of two natural numbers, a and b, is the largest natural number d such that $d \mid a$ and $d \mid b$. d is written as g.c.d. (a,b).

In chapter 2 you found g.c.d. (24, 16) essentially as follows:

$$A = \{1, 2, 3, 4, 6, 8, 12, 24\}$$

The set of factors of 16 we will call B:

$$B = \{1, 2, 4, 8, 16\}$$

Then

$$A \cap B = \{1, 2, 4, 8\}$$

is the set of common factors (divisors) of 16 and 24. Clearly 8 is the greatest common divisor of 24 and 16. That is, g.c.d. (24, 16) = 8. We see that 8 is the greatest natural number such that $8 \mid 24$ and $8 \mid 16$.

Question: Why will 1 always be an element in the intersection of the factor sets of two natural numbers?

A second solution to the above problem is as follows: By the unique factorization property of natural numbers we know that both 24 and 16 can be expressed as a product of primes where the factors of the product are unique. In fact we have $24 = 2 \cdot 2 \cdot 2 \cdot 3$ and $16 = 2 \cdot 2 \cdot 2 \cdot 2$. We see that the product expression $2 \cdot 2 \cdot 2$ is common to both factorizations and yields the greatest common divisor 8. This technique is useful when the numbers are small. For example, to find g.c.d. (45, 108) we determine that

$$45 = 3^2 \cdot 5$$

$$\text{and } 108 = 2^2 \cdot 3^2$$

We see that $3^2 = 9$ is the greatest common divisor of 45 and 108.

11.8 Exercises

1. Factor the numbers listed in as many ways as possible using

only two factors each time. We shall say that 2·3 is not different from 3·2 because of the commutative property of multiplication in (N, ·).

(a) 9
(b) 10
(c) 15
(d) 100
(e) 24
(f) 16
(g) 72
(h) 81

2. Write a complete factorization of

(a) 9
(b) 10
(c) 15
(d) 100
(e) 24
(f) 16
(g) 81
(h) 210
(i) 200
(j) 500

3. What factors of 72 do not appear in a complete factorization of 72?

4. What will be true about the complete factorization of every
(a) even natural number?
(b) odd natural number?

5. Construct at least two factor trees for each of the following:

(a) 24
(b) 96
(c) 625
(d) 1000

6. Find the greatest common divisor of the following pairs of numbers by making use of their complete factorizations:

(a) 70 and 90
(b) 80 and 63

(c) 372 and 90

(d) 663 and 1105

7. Determine if g.c.d. is a binary operation on N. If it is, explore its properties. If it fails to be a binary operation on N, explain why it fails.

8. Copy the following table for natural numbers and complete it through n = 30.

n	Factors of n	Number of factors	Sum of factors
1	1	1	1
2	1,2	2	3
3	1,3	2	4
4	1,2,4	3	7
5	1,5	2	6
6	1,2,3,6	4	12
7	1,7	2	8
8	1,2,4,8	4	15

(a) Which numbers represented by *n* in the table above have exactly two factors?

(b) Which numbers *n* have exactly three factors?

(c) If $n = p^2$ (where *p* is a prime number), how many factors does *n* have?

(d) If n = pq (where *p* and *q* are prime numbers and not the same), how many factors does *n* have? What is the sum of its factors?

(e) If $n = 2^k$ (where *k* is a natural number), how many factors does *n* have?

(f) If $n = 3^k$ (where *k* is a natural number and *p* is a prime), how many factors does *n* have?

(g) If $n = p^k$ (where *k* is a natural number and *p* is a prime), how many factors does *n* have?

(h) Which numbers n have $2n$ for the sum of their factors? (These numbers are called *perfect numbers*.)

9. If we list the set of multiples of 30, we obtain {30, 60, 90, 120, 150, 180, ...}. Also, if we list the set of multiples of 45, we obtain {45, 90, 135, 180, 225, 270, ...}. We see that a common multiple of 30 and 45 is 180. However, there is a common multiple which is the least common multiple of 30 and 45; namely 90. We write this as l.c.m. (30,45) = 90

(a) Examine the complete factorizations of 30 and 45 and explain how one could use these to find that the least common multiple of 30 and 45 is 90.

(b) Similarly, find the least common multiples of the following pairs of numbers by making use of their complete factorizations:

(1) 30 and 108
(2) 45 and 108
(3) 15 and 36
(4) 81 and 210
(5) 16 and 24
(6) 200 and 500

(c) Can you find any relationship between the greatest common divisor (g.c.d.) of a and b and the least common multiple (l.c.m.) of the same a and b? Experiment and write a report on your findings.

10. Determine if l.c.m. is a binary operation on N. Write a report of your findings.

11.9 The Sieve of Eratosthenes

The fact that every composite number can be expressed as a product of primes in one and only one way, except for order, indicates that the prime numbers are the basic elements, the atoms so to speak, in the structure of the natural numbers by multiplication. If we wish to have a basic understanding of multiplication of natural numbers (and division, which is defined in terms of multiplication), then it is to our advantage to be aware of some properties of the set of prime numbers.

A list of all the primes up to a given natural number N may be constructed as follows: Write down in order all the natural numbers less than N. We have done this below for N = 52. Then strike out 1 because by definition it is not a prime. Next, encircle 2 because it is a prime number. Then strike out all remaining multiples of 2 in the list, that is, 4, 6, 8, 10, etc. Such multiples of 2 are, as we discussed earlier, composite numbers.

Next encircle 3, the next number we encounter in our list. After 3 is encircled, we strike out 6, 9, 12, ..., that is, all multiples of 3 remaining in the list. (Note that 6 was struck out when we considered multiples of 2.) In a similar way we continue this process by next encircling 5 and striking out its remaining multiples. Lastly we encircle 7 and strike out its remaining multiples.

~~1~~ ② ③ ~~4~~ ⑤ ~~6~~ ⑦ ~~8~~ ~~9~~ ~~10~~
⑪ ~~12~~ ⑬ ~~14~~ ~~15~~ ~~16~~ ⑰ ~~18~~ ⑲ ~~20~~
~~21~~ ~~22~~ ㉓ ~~24~~ ~~25~~ ~~26~~ ~~㉗~~ ~~28~~ ㉙ ~~30~~
㉛ ~~32~~ ~~33~~ ~~34~~ ~~35~~ ~~36~~ ㊲ ~~38~~ ~~39~~ ~~40~~
㊶ ~~42~~ ㊸ ~~44~~ ~~45~~ ~~46~~ ㊼ ~~48~~ ~~49~~ ~~50~~
~~51~~.

Note that if we encircle all the numbers remaining in the list we obtain all the prime numbers less than N = 52. In all there are 15 such prime numbers obtained by this process, known as the Sieve of Eratosthenes. The sieve catches all the primes less than N in its meshes.

Complete tables of all primes less than 10,000,000 have been computed by this method and refinements of this method. Such tables are useful in supplying data concerning the distribution and properties of the primes.

Even the small list constructed above gives some indication that the primes are not distributed in any sort of obvious way among the natural numbers. Also, we see that it may happen that a number, p, is a prime and p + 2 is also a prime. Such pairs of primes are called twin primes. Examples of twin primes in the list above include 11 and 13, 17 and 19, 29 and 31, 41 and 43.

11.10 Exercises

1. (a) In the above list, what was the first number struck out that had not previously been struck out when we sieved

for the following:

(1) multiples of 2 (3) multiples of 5

(2) multiples of 3 (4) multiples of 7

(b) Can you make a conjecture concerning the first number struck out if we sieve for multiples of a prime p?

(c) Explain why we did not have to sieve for multiples of the prime 11?

(d) What is true of all numbers that

(1) pass through the sieve?

(2) remain in the sieve?

(e) Would any new numbers be crossed out if we sieved for multiples of 4? Why or why not?

2. Make up a list of natural numbers less than 131.

(a) Carry out the Sieve of Eratosthenes process on this set of numbers.

(b) How many primes are there less than 101?

(c) How many primes are there less than 131?

(d) What is the largest prime number in your list?

(e) What is the largest prime, p, for which you had to determine multiples in the sieving process? Explain.

3. (a) List the pairs of prime numbers less than 100 which have difference of 2.

(b) What name is given to such pairs?

(c) How many such pairs are there less than 100?

4. Make up a list of numbers which goes from 280 through 290.

(a) Apply the Sieve of Eratosthenes process to this list.

(b) List all the primes obtained from this sieving.

(c) For which primes did you have to seek multiples?

(d) Explain why you selected a certain prime as the largest for which you sought multiples.

5. (a) List the triplets of prime numbers less than 131 in which the succeeding numbers differ by 2. Such triplets are called prime triplets.

(b) After you have found the smallest set of prime triplets, explain why no other distinct set of prime triplets could have 3 as a factor of one of its numbers.

(c) Assume that there is a second set of prime triplets. Call them p, p + 2, p + 4. From (b) we know that $p \neq 3k$ where k is some natural number larger than 1. Why?

(d) If $p \neq 3k$, then what is the remainder obtained when p is divided by 3?

(e) Can you examine p + 2 and p + 4 and prove that p, p + 2, and p + 4 are not all primes if $p > 3$?

(f) What conclusion can you draw from (a) - (e)?

11.11 On the Number of Primes

Euclid (circa 300 B.C.) answered the following question: Is there a finite or an infinite number of prime numbers? As you work with the sieve of Eratosthenes you probably note that as you continue sieving the primes become relatively scarce. However, Euclid proved that, as one continues to examine the set of natural numbers, primes will always be encountered if we seek long enough. He proved that there are an infinite number of primes.

Euclid's argument proceeds as follows: Assume there is a largest prime. Let us denote this largest prime as "P." All the primes can then be written in a finite sequence

$$2, 3, 5, 7, \ldots, P.$$

Since P is the largest prime, all numbers greater than P must be composite; that is, every number greater than P must be divisible by at least one of the primes in the above sequence. But now consider the number

$$N = (2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot P) + 1.$$

that is, the number obtained by adding 1 to the product of all the primes. Since N is greater than P, it must be a composite number, and therefore divisible by at least one of the primes in the above sequence. But by which? It can be argued that N is not divisible by any of the primes 2, 3, 5, 7, ..., P, since dividing N by any of the primes yields a remainder of 1. Hence N cannot have any prime factors, which contradicts the fact that N is composite. Therefore, the assumption that the number of primes is finite leads to a contradiction, and we must conclude that there are an infinite number of primes.

It is interesting to note that it is not known whether the number of prime twins if finite or infinite. Unlike the situation for the primes, efforts to determine whether the number of prime twins is finite or infinite have not proved successful.

Another famous unsolved problem also deals with primes. It is called Goldbach's Conjecture. Goldbach stated, in a letter to Euler in 1742, that in every case that he tried he found that any even number greater than 2 could be represented as the sum of two primes. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, etc. No one has yet been able to prove or disprove this conjecture of Goldbach. The problem posed in the conjecture is interesting because (1) it is easily stated and (2) it involves addition whereas primes are defined in terms of multiplication. In any case, it has resisted solution for over two hundred years.

11.12 Exercises

1. Show that the following numbers all satisfy Goldbach's conjecture:

   (a) 10
   (b) 12
   (c) 14
   (d) 16
   (e) 18
   (f) 20
   (g) 36
   (h) 48
   (i) 100
   (j) 240

2. In working with Euclid's proof that the set of primes is infinite we find that possible values of N include $2 + 1$, $2\cdot3\cdot + 1$, $2\cdot3\cdot5 + 1$, $2\cdot3\cdot5\cdot7 + 1$, $2\cdot3\cdot5\cdot7\cdot11 + 1$, $2\cdot3\cdot5\cdot7\cdot11\cdot13 + 1$, $2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17 + 1$.

   (a) Explain how each of the numbers in the above list was formed. In each case what is P? What is N?

(b) The first 5 numbers in the list are primes. Compute then and verify that at least 4 of them are in fact primes.

(c) Note that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031$ and this number is composite because $30031 = (59)(509)$. Verify this.

(d) Prove that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1$ is a composite number. (Hint: Be efficient!)

(e) Discuss Euclid's argument with regard to the number shown in (d).

(f) Explain why a computer could never settle the question concerning the number of twin primes.

11.13 Euclid's Algorithm

We have seen that one way to find the g.c.d. of two natural numbers is to begin by expressing each of the numbers as a product of prime factors. However, this is not practical when the numbers considered are quite large. A method which is often used to find the g.c.d. of two large numbers is based on repeated use of the division algorithm.

We illustrate this by considering the problem of finding the g.c.d. of 28 and 16. By applying the division algorithm we have

$$28 = (1 \cdot 16) + 12 \text{ where } 0 \leq 12 < 16.$$

Note that if $a \mid (b + c)$ and $a \mid b$, then $a \mid c$. Thus any number that divides 28 and 16 must also divide 12. Thus the g.c.d. (28, 16) must divide 12. Let g.c.d. $(28, 16) = d$. Then $d \mid 12$

implies d is a common divisor of 16 and 12.

Also

$$d = \text{g.c.d. } (16, 12)$$

because if there was a larger divisor of 16 and 12 it would divide 28 and then d would not be g.c.d. (28, 16). Hence, we have g.c.d. (28, 16) = g.c.d. (16, 12). We continue the process by using the division algorithm again to obtain

$$16 = (1 \cdot 12) + 4 \text{ where } 0 \leq 4 < 12$$

By the same argument as above we have g.c.d. (16, 12) = g.c.d. (12, 4). Therefore, g.c.d. (28, 16) = g.c.d. (12, 4). Lastly, we apply the division algorithm to obtain

$$12 = (3 \cdot 4) + 0$$

and we see that the g.c.d. (12, 4) = 4

Thus g.c.d. (28, 16) = 4.

The following example illustrates the algorithm indicated above:

Example: Find the g.c.d. of 7469 and 2387

$$7469 = 2387 \cdot 3 + 308$$

g.c.d. (7469, 2387) = g.c.d. (2387, 308)

$$2387 = 308 \cdot 7 + 231$$

g.c.d. (2387, 308) = g.c.d. (308, 231)

$$308 = 231 \cdot 1 + 77$$

g.c.d. (308, 231) = g.c.d. (231, 77)

$$231 = 77 \cdot 3 + 0$$

g.c.d. (231, 77) = 77

Thus g.c.d. (7469, 2387) = 77

Note that we first divide the larger number, 7469, by the smaller number, 2387, and find the remainder 308 (which is less than the smaller number). Next we divide the smaller number by this remainder 308 and find a new remainder 231. Now we divide the first remainder 308 by the new remainder 231 and find the third remainder, 77. We continue this division until we obtain a remainder 0. The last non-zero remainder thus found is the g.c.d.

The procedure of computing the g.c.d. by successive applications of the division algorithm is known as Euclid's Algorithm.

It can happen that when we find the g.c.d. of two numbers it turns out to be 1. For example, it is clear that

$$\text{g.c.d. } (5, 13) = 1$$

and with a little work we can see that

$$\text{g.c.d. } (124, 23) = 1$$

Such pairs of numbers whose g.c.d. is 1 play an important role in Number Theory.

Definition 8: If the greatest common divisor of two natural numbers a and b is 1, we say that a and b are relatively prime.

Thus 5 and 13 are relatively prime since g.c. d. (5, 13) = 1. Similarly 124 and 23 are relatively prime. We shall use the idea of two numbers being relatively prime in our next axiom.

A8. If d = g.c.d. (a,b), then there exist integers x and y: such that

$$d = x \cdot a + y \cdot b$$

In particular, if a and b are relatively prime, there exist integers x and y such that $1 = x \cdot a + y \cdot b$.

Example 1: g.c.d. (72, 86) = 2 and 2 = 6(72) + (-5)(86). Here x = 6 and y = -5.

Example 2: g.c.d. (7, 5) = 1 and 1 = 3(7) + (-4)(5).
Here x = 3 and y = -4.

Example 3: g.c.d. (147, 130) = 1 and
1 = 23 (147) + (-26)(130).
Here x = 23 and y = -26.

We will use A8 next to prove an important theorem which will enable us to prove a number of other theorems that tie together the ideas of "prime" and "divisibility."

Theorem: If a | bc and g.c.d. (a,b) = 1, then a | c.

Proof: Since g.c.d. (a,b) = 1, then, by A8 there are integers x and y, such that 1 = ax + by
Since c = c we have by theorem A, c·1 = c·(ax + by).
Applying A6 on the left and A5 on the right, we have:

$$c = cax + cby$$

By hypothesis a |bc which by A3 implies
a | c b. But a | cb implies a | cby. (Why?)
Similarly a | cax. Thus, we conclude that a | c.
(Why?)

Example 1: 7 | 70. Consider 70 as 5·14. Then we have
7 | (5·14) and g.c.d. (7,5) = 1. Hence by the above theorem 7 | 14.

Example 2: 10 | 840. Consider 840 as 21·40. Then we have
10 | (21·40) and g.c.d. (10,21) = 1. Hence
10 | 40.

Among the theorems that are easily established using the above theorem are:

(1) Let p be a prime such that $p \mid (bc)$ and $p \nmid b$. Then $p \mid c$.

(2) If p is a prime and $p \mid ab$, then either $p \mid a$ or $p \mid b$ (or both).

11.14 Exercises

1. Using the Euclidean Algorithm find the greatest common divisor of each of the following pairs of numbers:

   (a) 1122 and 105 (c) 220 and 315

   (b) 2244 and 418 (d) 912 and 19,656

2. Find the g.c.d. (144, 104) using two different methods.

3. (a) What is the g.c.d. of a and b if a and b are distinct primes?

   (b) If a is a prime and b is a natural number such that $a \mid b$ what is the g.c.d. (a,b)?

4. Prove the following: Let p be a prime such that $p \mid (bc)$ and $p \nmid b$. Then $p \mid c$.

5. Prove: If p is a prime and $p \mid ab$ then either $p \mid a$ or $p \mid b$ (or both).

6. Prove: If a and b are relatively prime and $a \mid c$ and $b \mid c$, then $ab \mid c$.

7. Prove: If d = g.c.d. (a,b) and a = rd and b = sd, then r and s are relatively prime.

8. Find 7 consecutive natural numbers each of which is composite. Find 8. (It can be proved that there are a million consecutive composite natural numbers; in fact, any number, no matter how large.)

9. Fermat's Little Theorem. In the year 1640 Fermat stated the following: If p is a prime that is not a divisor of the natural number a, then $p \mid (a^{p-1} - 1)$.

   (a) Find two examples which illustrate this theorem.

   (b) Note that there is the restriction that $p \nmid a$. What would follow if $p \mid a$?

   (c) What can we conclude if p is not a prime?

   (d) Can you prove Fermat's Little Theorem?

11.15 Summary

In this chapter we have explored topics in number theory. You have had an opportunity to make conjectures and then to prove your conjectures.

At this time you should be able to give a clear description of what is meant by factor, multiple, prime number, composite number, even and odd natural numbers, greatest common divisor, least common multiple, and complete factorization. Can you state the Unique Factorization Property of the natural numbers? You saw that the Sieve of Eratosthenes provides one way to determine primes up to some number. Do you believe that this is an efficient tool for finding primes? Can you describe several ways of finding the g.c.d. of two natural numbers? What purpose did Euclid's Algorithm serve and on what principle was it based? Can you state some properties of prime numbers? Can you state some problems that no one has yet been able to solve?

Overall, your awareness of the set of natural numbers should be increased. Also you should be more aware of what constitutes a proof in mathematics and the fact that there are different methods of proving theorems.

11.16 Review Exercises

1. Explain why the following are true.
   - (a) 10 is a factor of 50.
   - (b) 30 is a multiple of 6.
   - (c) 6 is a factor of 30.
   - (d) 6 is a factor of 6.
   - (e) 7 is not a factor of 30.
   - (f) 7 is a prime number.
   - (g) 6 is a composite number.
   - (h) 91 is a composite number.
2. Define the following terms:
   - (a) factor
   - (c) prime
   - (b) multiple
   - (d) composite
3. Give a complete factorization of each of the following:
   - (a) 38
   - (c) 96
   - (b) 72
   - (d) 97
4. Using the data in 3 above, determine:
   - (a) g.c.d. (38, 72)
   - (c) g.c.d. (72, 96)
   - (b) g.c.d. (38, 96)
   - (d) g.c.d. (72, 97)
5. Using the data obtained in 3, determine:
   - (a) l.c.m. (38, 72)
   - (c) l.c.m. (72, 96)
   - (b) l.c.m. (38, 96)
   - (d) l.c.m. (72, 97)

6. Using the Sieve of Eratosthenes, determine all primes between 130 and 150.
   (a) How many primes are in this set of numbers?
   (b) How many twin primes are in this set?
   (c) What is the largest prime p for which you have to determine multiples to find all the primes in this set of numbers?
7. Using the Euclidean algorithm check your answer for 4 (c) above.
8. Prove: if a | b and b | c, then a | c where a, b, c ∈ N.
9. If 9 | n and 10 | n does it follow that 90 | n? Explain.
10. Prove if a ∤ b where a is a prime, then g.c.d. (a,b) = 1.

CHAPTER 12

THE RATIONAL NUMBERS

12.1 W, Z and Z_7

In earlier chapters we studied a variety of number systems --whole numbers, clock numbers, and integers. The operational systems $(W,+,\cdot)$, $(Z_7,+,\cdot)$ and $(Z,+,\cdot)$ have several important properties in common. In each system, + and $\cdot$ are associative and commutative, $\cdot$ is destributive over +, "0" represents the identity element for +, "1" represents the identity element for $\cdot$, and for any elements a in each system $a \cdot 0 = 0 \cdot a = 0$.

However, the differences among these systems are as striking as the similarities. Z_7 is finite; W and Z are infinite. The assignments made by + and $\cdot$ in Z_7 are quite different from those made by + and $\cdot$ in W and Z.

$(Z_7,+,\cdot)$	$(W,+,\cdot)$ and $(Z,+,\cdot)$
$4 + 3 = 0$	$4 + 3 = 7$
$6 + 6 = 5$	$6 + 6 = 12$
$5 \cdot 6 = 2$	$5 \cdot 6 = 30$

In $(W,+)$ subtraction is not an operation (What is $7 - 10$?), only 0 has an additive inverse $(0 + 0 = 0)$, and many simple equations of the type "$a + x = b$" have no solution (for example, $75 + x = 50$). In both $(Z_7,+)$ and $(Z+)$ subtraction is an operation, each element has an additive inverse, and all equations of the type "$a + x = b$" have solutions. The integers were developed specifically to meet these deficiencies in $(W,+)$. In $(Z,+)$ $7 - 10 = -3$, $a + (-a) = 0$ for every a, and the solution set of $75 + x = 50$ is $\{-25\}$.

Extension of W to Z removed many of the restrictions on addition and subtraction in W, but it did not accomplish the same purpose with respect to mulitplication and division. In both $(W,\cdot)$ and $(Z,\cdot)$ division is not an operation, (What is $7 \div 10$?), only 1 and -1 have multiplicative inverses ($1 \cdot 1 = 1 = (-1)(-1)$), and many simple equations of the form "$a \cdot x = b$" have no solution (for example, $75 \cdot x = 50$).

These limitations of mulitplication and division do not hold in $(Z_7,\cdot)$.

.	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

In Z_7 division is almost always possible. For example, $6 \div 2 = 3$ since $3 \cdot 2 = 6$; $5 \div 2 = 6$ since $6 \cdot 2 = 5$; $3 \div 5 = 2$ since $2 \cdot 5 = 3$. Only division by zero is not possible. If $3 \div 0 = m$, then $3 = m \cdot 0 = 0$. Clearly, $3 \neq 0$; so there is no number m in Z_7 for which $3 \div 0 = m$.

In $(Z_7,\cdot)$ every element -- except 0, of course -- has a multiplicative inverse. The multiplicative inverse of 2 (which was written "$\frac{1}{2}$" in Chapter one) is 4, since $4 \cdot 2 = 1$ in $(Z_7,\cdot)$. Similarly $\frac{1}{3} = 5$ since $5 \cdot 3 = 1$ in $(Z_7,\cdot)$; $\frac{1}{4} = 2$ since $2 \cdot 4 = 1$ in $(Z_7,\cdot)$; $\frac{1}{5} = 3$ since $3 \cdot 5 = 1$ in $(Z_7,\cdot)$; and $\frac{1}{6} = 6$ since $6 \cdot 6 = 1$ in $(Z_7,\cdot)$.

In $(Z_7, \cdot)$ any equation of the form "$a \cdot x = b$" has a solution, if $a \neq 0$. For example, the solution set of $5 \cdot x = 1$ is $\{3\}$ since $5 \cdot 3 = 1$ in $(Z_7, \cdot)$. The solution set of "$5 \cdot x = 2$" is $\{6\}$ since $5 \cdot 6 = 2$ in $(Z_7, \cdot)$. In Chapter one this last solution set was also written $\{\frac{2}{5}\}$. $\frac{2}{5} = 2 \div 5 = 6$ because $6 \cdot 5 = 2$ in $(Z_7, \cdot)$.

The fact that $(Z_7, +, \cdot)$ apparently has all the good properties of $(W, +, \cdot)$ and $(Z, +, \cdot)$ and none of the deficiencies makes it more desirable from a mathematical point of view. However Z_7 has its own peculiar drawbacks. For example, such a simple but important process as counting elements in a set is severely limited if $\{0,1,2,3,4,5,6\}$ is all that is available. It is impossible to use Z_7 for ordering sets into first, second, third, fourth,... ; what comes after the sixth in Z_7?

There is a broader class of practical problems for which none of Z_7, W, Z is adequate.

Example 1: What is the probability that a fair coin will turn up "heads" when tossed?

Example 2: A board to be used for a 5 shelf bookcase is 14 feet long. How long should each shelf be?

Example 3: An architect's drawing of a building uses a scale of 1 inch equals 4 feet. How long should a line segment on the drawing be if the segment represents an actual length of 15 feet?

These problems will never be solved using $(Z_7, +, \cdot)$ as a mathematical model. Certainly the elements of Z_7 cannot be used to measure lengths, for there are a large number of lengths in feet such as 8 feet, 25 feet, etc. that cannot be given by elements of Z_7. Of course, as we have seen, $(Z,+,\cdot)$ and $(W,+,\cdot)$ aren't much help either. The three situations mentioned call for solutions of the equations

$$(1)\quad 2 \cdot x = 1$$
$$(2)\quad 5 \cdot x = 14$$
$$(3)\quad 4 \cdot x = 15$$

which do not have integer solutions.

What is needed is an extension of $(Z,+,\cdot)$. In this extended number system (1), (2), (3), and all other equations of the type "$a \cdot x = b$," with $a \neq 0$ should have solutions.

We shall now begin the construction of such an extension. That is, we shall construct a number system $(Q,+,\cdot)$ such that $Z \subset Q$ and such that the following properties hold for $(Q,+,\cdot)$:

(1) Addition and multiplication are associative and commutative.

(2) The distributive law holds for multiplication over addition.

(3) 0 and 1 are the identity elements for addition and multiplication, respectively.

(4) Every element has an inverse under addition.

(5) Subtraction is an operation.

Furthermore, we want $(Q,+,\cdot)$ to have the following properties:

(6) Every equation "$ax = b$" where $a \in Q$ and $b \in Q$ and $a \neq 0$ has a solution in Q.

(7) Every member of Q (except 0) has a multiplicative inverse.

(8) Division (except by 0) is always possible.

12.2 Reciprocals of the Integers

The whole numbers were extended to the integers by uniting $\{0,1,2,3,\ldots\}$ and a new set $\{-1,-2,-3,-4,\ldots\}$ containing additive inverses for each non-zero whole number. This suggests that if the integers are combined with a new set $Z' = \{\frac{1}{1}, \frac{1}{-1}, \frac{1}{2}, \frac{1}{-2}, \frac{1}{3}, \frac{1}{-3}$ $\frac{1}{4}, \frac{1}{-4}, \ldots\}$ where $\frac{1}{a}$ (read: "1 over a") is the multiplicative inverse of integer a, where $a \neq 0$, the problems of division, multiplicative inverse, and equations "$a \cdot x = b$" would be solved.

How should addition and multiplication be defined in this new set $Z \cup Z'$? Clearly, if a and b are integers the product $a \cdot b$ should be computed as it is in $(Z, \cdot)$. Furthermore, if the multiplication properties of 0 and 1 are to hold in $Z \cup Z'$, $0 \cdot \frac{1}{a} = 0$ and $1 \cdot \frac{1}{a} = \frac{1}{a}$ for all $\frac{1}{a}$ in Z'. Thus $1 \cdot \frac{1}{3} = \frac{1}{3}$, $1 \cdot \frac{1}{-7} = \frac{1}{-7}$, and $0 \cdot \frac{1}{-3} = 0$. Since $\frac{1}{a}$ is the multiplicative inverse of a, $a \cdot \frac{1}{a} = 1$ for all a (except 0) in Z. For example, $5 \cdot \frac{1}{5} = 1$ and $-6 \cdot \frac{1}{-6} = 1$.

Question: What is $\frac{1}{a} \cdot 0$? $\frac{1}{a} \cdot 1$? $\frac{1}{a} \cdot a$? Can you explain your answer?

What element of $Z \cup Z'$ should be assigned as the product of $\frac{1}{2}$ and $\frac{1}{3}$? of $\frac{1}{-2}$ and $\frac{1}{3}$? of $\frac{1}{-2}$ and $\frac{1}{-3}$? We know that $2 \cdot \frac{1}{2} = 1$ and $3 \cdot \frac{1}{3} = 1$. Therefore,

$$1 \cdot 1 = (2 \cdot \tfrac{1}{2})(3 \cdot \tfrac{1}{3}) = 1.$$

Since multiplication in $Z \cup Z'$ should be commutative and associative,

$$\begin{aligned} 1 &= (2 \cdot \tfrac{1}{2})(3 \cdot \tfrac{1}{3}) \\ &= (2 \cdot 3)(\tfrac{1}{2} \cdot \tfrac{1}{3}). \end{aligned}$$

Thus the product $\frac{1}{2} \cdot \frac{1}{3}$ must be the multiplicative inverse of the product $2 \cdot 3 = 6$. Therefore $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2 \cdot 3}$.

The fact that

$$\begin{aligned} 1 &= (-2 \cdot \tfrac{1}{-2})(3 \cdot \tfrac{1}{3}) \\ &= (-2 \cdot 3)(\tfrac{1}{-2} \cdot \tfrac{1}{3}) \\ &= -6 \cdot (\tfrac{1}{-2} \cdot \tfrac{1}{3}) \end{aligned}$$

implies that $(\frac{1}{-2} \cdot \frac{1}{3}) = \frac{1}{-6} = \frac{1}{-2 \cdot 3}$. Similarly, $\frac{1}{-2} \cdot \frac{1}{-3} = \frac{1}{6} = \frac{1}{(-2)(-3)}$ and, in general, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$. But what element of $Z \cup Z'$ is assigned as the product of a and $\frac{1}{b}$? For example, what is $5 \cdot \frac{1}{3}$ in $Z \cup Z'$? What is $7 \cdot (\frac{1}{-8})$? What is $(-3)(\frac{1}{-5})$?

The number assigned as the product of 5 and $\frac{1}{3}$ must be a solution of the equation "$3 \cdot x = 5$," since

$$\begin{aligned} 3 \cdot (5 \cdot \tfrac{1}{3}) &= 3 \cdot (\tfrac{1}{3} \cdot 5) \\ &= (3 \cdot \tfrac{1}{3}) \cdot 5 \\ &= 1 \cdot 5 \\ &= 5. \end{aligned}$$

But "$3 \cdot x = 5$" has no integer solution, and there is no multiplicative inverse of an integer which satisfies this equation. Therefore it does not seem possible to make $(Z \cup Z', \cdot)$ into an operational system which retains the structure and properties of $(Z,+,\cdot)$. Our hope of obtaining an extension of Z that has the

8 properties listed at the end of Section 12.1, simply by appending a set of multiplicative inverses to Z was in vain. Another extension must be made, from $Z \cup Z'$ to a new set Q.

12.3 Exercises

Exercises 1 - 7 refer to $(Z_7, \cdot)$.

1. If possible, give another name for:

   (a) $\frac{1}{2}$
   (b) $\frac{1}{3}$
   (c) $\frac{1}{4}$
   (d) $\frac{1}{5}$
   (e) $\frac{1}{6}$
   (f) $\frac{1}{0}$

2. Compute:

   (a) $3 \cdot \frac{1}{2}$
   (b) $5 \cdot \frac{1}{3}$
   (c) $4 \cdot \frac{1}{2}$
   (d) $4 \cdot \frac{1}{5}$
   (e) $6 \cdot \frac{1}{5}$
   (f) $6 \cdot \frac{1}{3}$
   (g) $2 \cdot \frac{1}{3}$
   (h) $2 \cdot \frac{1}{5}$

3. In $(Z_7, \cdot)$ $\frac{a}{b} = a \div b = c$ if and only if $a = c \cdot b$. For example, $\frac{3}{4} = 3 \div 4 = 6$ because $3 = 6 \cdot 4$.

   Compute:

   (a) $\frac{3}{2}$
   (b) $\frac{5}{3}$
   (c) $\frac{4}{2}$
   (d) $\frac{4}{5}$
   (e) $\frac{6}{5}$
   (f) $\frac{6}{3}$
   (g) $\frac{2}{3}$
   (h) $\frac{2}{5}$

4. Solve:

(a) $2x = 3$
(b) $3x = 5$
(c) $2x = 4$
(d) $5x = 4$
(e) $5x = 6$
(f) $3x = 6$
(g) $3x = 2$
(h) $5x = 2$

5. Solve:

(a) $2x = 1$
(b) $3x = 1$
(c) $4x = 1$
(d) $5x = 1$
(e) $6x = 1$
(f) $0x = 1$

6. Compare your answers to Exercises 2, 3 and 4 and explain the pattern that you notice.

7. Compare your answers to Exercises 1 and 5 and explain the pattern that you notice.

Exercises 8 - 12 refer to $(Z \cup Z', \cdot)$

8. In $(Z \cup Z', \cdot)$ the number x is a multiplicative inverse of y if $x \cdot y = 1$. Find, if possible, a multiplicative inverse of each of the following elements of $Z \cup Z'$.

(a) -7
(b) 13
(c) $\frac{1}{17}$
(d) $\frac{1}{-11}$
(e) 0
(f) 1
(g) $\frac{1}{-18}$
(h) -1

9. Compute:

(a) $(104)(-1)$
(b) $(-8) \cdot (13)$
(c) $(52) \cdot (-2)$
(d) $(4) \cdot (-26)$

(e) $(-13) \cdot (8)$

(f) $\frac{1}{5} \cdot \frac{1}{7}$

(g) $\frac{1}{-5} \cdot \frac{1}{-7}$

(h) $\frac{1}{4} \cdot \frac{1}{-9}$

(i) $\frac{1}{-4} \cdot \frac{1}{9}$

(j) $\frac{1}{9} \cdot \frac{1}{-4}$

(k) $(\frac{1}{3} \cdot \frac{1}{-5}) \cdot \frac{1}{-10}$

(l) $\frac{1}{3} \cdot (\frac{1}{-5} \cdot \frac{1}{-10})$

(m) $(\frac{1}{-3} \cdot \frac{1}{5}) \cdot \frac{1}{-10}$

(n) $(\frac{1}{-4} \cdot \frac{1}{-8}) \cdot \frac{1}{-6}$

(o) $\frac{1}{-8} \cdot (\frac{1}{-6} \cdot \frac{1}{-4})$

10. In $(Z,\cdot)$ $12 = 4 \cdot 3 = 3 \cdot 4 = 6 \cdot 2 = 2 \cdot 6 = 12 \cdot 1 = 1$
$1 \cdot 12 = (-4)(-3) = \ldots$

Write each of the following integers in three ways as products of integers.

(a) 9

(b) 75

(c) -15

11. $\frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{-3} \cdot \frac{1}{-2} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{-2} \cdot \frac{1}{-3}$. Write each of the following elements of $Z \cup Z'$ in three ways as products.

(a) $\frac{1}{12}$

(b) $\frac{1}{75}$

(c) $\frac{1}{-15}$

12. Although 1 and -1 have multiplicative inverses 1 and -1 respectively in Z, it is convenient to consider them as elements of Z', the set of inverses of integers. Thus $1 = \frac{1}{1}$ and $-1 = \frac{11}{-1}$.

Compute:

(a) $\frac{1}{5} \cdot \frac{1}{1}$

(b) $\frac{1}{-12} \cdot \frac{1}{1}$

(c) $\frac{1}{1} \cdot \frac{1}{-17}$

(d) $\frac{1}{1} \cdot \frac{1}{4}$

(e) $\frac{1}{-1} \cdot \frac{1}{5}$

(f) $\frac{1}{-1} \cdot \frac{1}{-12}$

(g) $\frac{1}{-17} \cdot \frac{1}{-1}$

(h) $\frac{1}{4} \cdot \frac{1}{-1}$

13. For each of the following equations, give the solution in the set Z of integers.

(a) $x + 4 = 6$

(b) $x + 4 = 1$

(c) $312 + x = 298$

(d) $500 + x = -6$

(e) $6 + x = 0$

(f) $x + 2000 = 0$

(g) $15 + x = 25$

(h) $15 + x = -25$

(i) $-330 + x = 45$

(j) $-330 + x = -45$

(k) $-20 + x = -100$

(l) $x + 1,215,687 = 1,200,347$

14. For each of the following equations, list the set of integers which are solutions. If this set is empty, say so.

(a) $-3 \cdot x = -21$

(b) $-3 \cdot x = 21$

(c) $-3 \cdot x = 20$

(d) $x \cdot 5 = 45$

(e) $x \cdot 5 = 102$

(g) $0 \cdot x = -2$

(h) $88x = 8800$

(i) $88x = -8800$

(j) $467x = 1401$

(k) $-12x = 144$

(f) $4 \cdot x = 0$

(l) $1367 = -7x$

15. Give an integer for each of the quotients below. If there is no integer, say so.

(a) $-21 \div -3$

(b) $21 \div -3$

(c) $20 \div -3$

(d) $-50 \div 5$

(e) $0 \div -4$

(f) $1 \div 84$

(g) $-42 \div 14$

(h) $-2313 \div -9$

(i) $-1000 \div 2000$

(j) $2103 \div (-3)$

(k) $27{,}521 \div -13$

16. In $(Z,+,\cdot)$, how many solutions are there for the equation "$0 \cdot x = 0$"?

17. If division is defined in $(Z',\cdot)$ by $\frac{1}{a} \div \frac{1}{b} = \frac{1}{c}$ if $\frac{1}{a} = \frac{1}{c} \cdot \frac{1}{b}$, can you find $\frac{1}{2} \div \frac{1}{3}$ in Z'? Can you find any quotients in Z'?

12.4 Extending $Z \cup Z'$ to Q

Extension of Z to $Z \cup Z'$ removed one limitation of Z; each number in $Z \cup Z'$ (except 0) has a multiplicative inverse. However equations such as "$5x = 3$" have no solutions in $Z \cup Z'$, and multiplication is not even an operation in $Z \cup Z'$ (what is $3 \cdot \frac{1}{5}$ in $Z \cup Z'$?). Can $Z \cup Z'$ be extended to a number system without these restrictions?

The system Q which meets the above requirements must contain $Z = \{0,1,-1,2,-2,3,-3,\ldots\}$ and $Z' = \{\frac{1}{1}, \frac{1}{-1}, \frac{1}{2}, \frac{1}{-2}, \frac{1}{3}, \frac{1}{-3},\ldots\}$. We have already shown how to assign products to (a,b) for $\underline{a}$ and $\underline{b}$ in Z, and to $(\frac{1}{a},\frac{1}{b})$, for $\frac{1}{a}$ and $\frac{1}{b}$ in Z'. Q must also contain numbers which can be assigned as products for pairs $(b,\frac{1}{a})$ of elements in Z ∪ Z'. Recall from Chapter 2 that for $\cdot$ to be an operation on Q it must assign exactly one element of Q to each pair of elements of Q.

Question: How shall we assign exactly one element of Q to each pair $(b, \frac{1}{a})$ of elements of Q?

If we agree to indicate the product $b \cdot \frac{1}{a}$ by "$\frac{b}{a}$" (read: "b over a"), then Q must contain $2 \cdot \frac{1}{3} = \frac{2}{3}$, $-7 \cdot \frac{1}{5} = \frac{-7}{5}$, $-6 \cdot \frac{1}{-11} = \frac{-6}{-11}$, and so on. The equation $5x = -3$ has a solution $\frac{-3}{5}$ in Q because

$$\begin{aligned} 5 \cdot \frac{-3}{5} &= 5 \cdot (-3 \cdot \frac{1}{5}) \\ &= -3 \cdot (5 \cdot \frac{1}{5}) \\ &= -3. \end{aligned}$$

Question: Why don't we worry about pairs $(\frac{1}{a}, b)$?

Question: What is the solution set in Q of "$4x = 3$"?

The natural answer to this question is $\{\frac{3}{4}\}$ since

$$\begin{aligned} 4(\frac{3}{4}) &= 4 \cdot (3 \cdot \frac{1}{4}) \\ &= 3 \cdot (4 \cdot \frac{1}{4}) \\ &= 3. \end{aligned}$$

But what about $\frac{6}{8}$, the product of 6 and $\frac{1}{8}$?

$$\begin{aligned} 4(\frac{6}{8}) &= 4 \cdot (6 \cdot \frac{1}{8}) \\ &= (4 \cdot 6) \cdot \frac{1}{8} \\ &= (3 \cdot 8) \cdot \frac{1}{8} \\ &= 3 \cdot (8 \cdot \frac{1}{8}) \\ &= 3. \end{aligned}$$

What about $\frac{9}{12}$, the product of 9 and $\frac{1}{12}$?

$$\begin{aligned} 4 \cdot (\tfrac{9}{12}) &= 4 \cdot (9 \cdot \tfrac{1}{12}) \\ &= (4 \cdot 9)(\tfrac{1}{12}) \\ &= (3 \cdot 12)(\tfrac{1}{12}) \\ &= 3. \end{aligned}$$

In a similar way you can check that $\frac{12}{16}, \frac{15}{20}, \frac{18}{24}, \frac{21}{28}, \ldots, \frac{3n}{4n}, \ldots$ are all solutions of the same equation. Do these represent different elements of Q? If so, the equation $4x = 3$ has an infinite number of solutions--not a desirable state of affairs. (Why not?) If a and b are two different solutions of $4x = 3$ then $4a = 3$ and $4b = 3$, and so $4a = 4b$ and $a \neq b$. Thus the cancellation property, so useful in Z, would not carry over to Q.

Actually the situation is much simpler than it appears. "$\frac{3}{4}$", "$\frac{6}{8}$", "$\frac{9}{12}$", "$\frac{12}{16}$", ..., "$\frac{3n}{4n}$", ... all represent the same element of Q. In other words, the pairs $(3,\frac{1}{4})$, $(6,\frac{1}{8})$, $(9,\frac{1}{12})$, $(12,\frac{1}{16})$, ... are all assigned the same product in Q. This is easy to show. For instance,

$$\begin{aligned} 6 \cdot \tfrac{1}{8} &= (3 \cdot 2)(\tfrac{1}{2} \cdot \tfrac{1}{4}) \\ &= [3 \cdot (2 \cdot \tfrac{1}{2})] \cdot \tfrac{1}{4} \\ &= [3 \cdot 1] \cdot \tfrac{1}{4} \\ &= 3 \cdot \tfrac{1}{4} \end{aligned}$$

Therefore $\frac{6}{8} = \frac{3}{4}$ or "$\frac{6}{8}$" and "$\frac{3}{4}$" name the same number in Q. Similarly $\frac{9}{12} = \frac{3}{4}$, since

$$\begin{aligned}\frac{9}{12} &= 9 \cdot \frac{1}{12}\\ &= (3 \cdot 3)(\frac{1}{3} \cdot \frac{1}{4})\\ &= 3 \cdot (3 \cdot \frac{1}{3}) \cdot \frac{1}{4}\\ &= 3 \cdot \frac{1}{4}\\ &= \frac{3}{4}\ .\end{aligned}$$

The fact that single element of Q has an infinite number of names should not be shocking: $\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots$ are all products of elements in $Z \cup Z'$. In $(Z,\cdot)$ many pairs of integers are assigned the same integer as a product. For example, $12 = 4 \cdot 3$, $12 = 6 \cdot 2$, $12 = (-6)(-2)$, and so on. In $(Z,+)$ each integer can be obtained as a sum in an infinite number of ways. For example: $12 = 10 + 2$, $12 = 11 + 1$, $12 = 12 + 0$, $12 = 13 + (-1)$, $12 = 14 + (-2)$, and so on.

Looking again at the equation "$4x = 3$," try $\frac{-3}{-4}$ as a solution.

$$\begin{aligned}4 \cdot (\tfrac{-3}{-4}) &= 4 \cdot (-3 \cdot \tfrac{1}{-4})\\ &= (4 \cdot -3) \cdot \tfrac{1}{-4}\\ &= (3 \cdot -4) \cdot \tfrac{1}{-4}\\ &= 3 \cdot (-4 \cdot \tfrac{1}{-4})\\ &= 3\end{aligned}$$

Next try $\frac{-6}{-8}, \frac{-9}{-12}, \frac{-12}{-16}, \ldots$. Following the procedure illustrated you will find that these products also satisfy the equation. The reason is quite simple.

$$\begin{aligned} \frac{-3}{-4} &= \frac{3 \cdot (-1)}{4 \cdot (-1)} \\ &= (3 \cdot (-1)) \cdot (\tfrac{1}{4} \cdot \tfrac{1}{-1}) \\ &= 3 \cdot \tfrac{1}{4} \\ &= \tfrac{3}{4} . \end{aligned}$$

The pairs $(3,\frac{1}{4})$ and $(-3,\frac{1}{-4})$, $(-6,\frac{1}{-8})$, $(-9,\frac{1}{-12})$,... are all assigned the same product in Q.

The investigation of $3 \cdot \frac{1}{4}$ and $4x = 3$ can be repeated with any other product $a \cdot \frac{1}{b}$ and the corresponding equation $bx = a$. For instance, the pairs

$$(2,\tfrac{1}{3}),\ (-2,\tfrac{1}{-3}),\ (4,\tfrac{1}{6}),\ (-4,\tfrac{1}{-6}),\ldots,\ (2n,\tfrac{1}{3n}),\ldots$$

are all assigned the same product in Q. In other words,

$$\frac{2}{3} = \frac{-2}{-3} = \frac{4}{6} = \frac{-4}{-6} = \ldots = \frac{2n}{3n}$$

Reasoning as above, you should be able to convince yourself that

$$\frac{7}{-8} = \frac{-7}{8} = \frac{14}{-16} = \frac{21}{-24} = \frac{-21}{24} = \ldots = \frac{-7n}{8n} \ldots,$$

and

$$\frac{9}{10} = \frac{-9}{-10} = \frac{18}{20} = \frac{-18}{-20} = \ldots = \frac{9n}{10n} \ldots .$$

The element of Q assigned as the product of the pair $(a,\frac{1}{b})$ is also assigned as the product of $(na,\frac{1}{nb})$, where n is any integer (except 0). In other words, this element of Q can be named in an infinite number of ways!

"$\frac{a}{b}$", "$\frac{-a}{-b}$", "$\frac{2a}{2b}$", "$\frac{-2a}{-2b}$", ..., "$\frac{na}{nb}$" ... .

The elements of Q -- including integers, inverse of integers, and products -- are called *rational numbers*. The names of these rational numbers "$\frac{a}{b}$" are called *fractions*, each rational number having many *fraction names*. In the fraction "$\frac{a}{b}$" "a" is called the *numerator* and "b" the *denominator*.

When two fractions name the same rational number they are called *equivalent fractions*. For example $\frac{6}{8}$ and $\frac{-9}{-12}$ are equivalent fractions, as are $\frac{-2}{4}$ and $\frac{2}{-4}$, as are $\frac{3}{-4}$ and $\frac{-3}{4}$. It is easy to verify the following test for equivalence of fractions.

Let $b = 0$, $d = 0$. Then the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are *equivalent* if, and only if, $ad = bc$.

Question: Is equivalence of fractions an equivalence relation?

Because "$\frac{3}{4}$", "$\frac{-3}{-4}$", "$\frac{6}{8}$", and "$\frac{12}{16}$" all name the same rational number, when we say "the rational number $\frac{3}{4}$" or "the rational number $\frac{6}{8}$" or "the rational number $\frac{-3}{-4}$ " we refer to the same element of Q.

One of these fractional names, "$\frac{3}{4}$", is the simplest and most commonly used. "$\frac{3}{4}$" represents the product of the pair $(3,\frac{1}{4})$; the integers 3 and 4 have no common factor other than 1. On the other hand, "$\frac{6}{8}$" represents the product of the pair $(6,\frac{1}{8})$. But the integers 6 and 8 have a common factor, 2:

$$\begin{aligned}\frac{6}{8} &= \frac{2 \cdot 3}{2 \cdot 4} \\ &= (2 \cdot 3)(\tfrac{1}{2} \cdot \tfrac{1}{4}) \\ &= (2 \cdot \tfrac{1}{2})(3 \cdot \tfrac{1}{4}) \\ &= 1 \cdot (\tfrac{3}{4}) \\ &= \tfrac{3}{4}\end{aligned}$$

In a similar manner "$\frac{12}{16}$" can be simplified because 12 and 16 have a common factor, 4. "$\frac{3}{4}$" is called an irreducible fraction. (Note: It is the fraction "$\frac{3}{4}$" which is irreducible, not the rational number $\frac{3}{4}$!)

Example 1: $\frac{7}{8} = \frac{14}{16} = \frac{-21}{-24}$, but "$\frac{7}{8}$" is the irreducible fraction.

Example 2: $\frac{6}{9} = \frac{-2}{-3} = \frac{2}{3}$, but "$\frac{2}{3}$" is the irreducible fraction (not "$\frac{-2}{-3}$").

Example 3: $\frac{-5}{8} = \frac{5}{-8} = \frac{-10}{16}$, but "$\frac{-5}{8}$" is the irreducible fraction (preferred for reasons of convenience over "$\frac{5}{-8}$").

Example 4: $2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{-2}{-1}$, but we agree to take $\frac{2}{1}$ as the irreducible fraction. Similarly, we will agree that $\frac{-2}{1}$, $\frac{1}{1}$, $\frac{-1}{1}$ are irreducible fractions, not -2, 1 or -1.

The set of rational numbers, Q, promises to be the extension of Z required to solve equations "$a \cdot x = b$" and make division an operation. These questions are examined in Section 12.6.

12.5 **Exercises:**

1. Write four other fraction names for each of the following rational numbers:
   (a) $\frac{2}{3}$
   (b) $\frac{-6}{-8}$
   (c) $\frac{-3}{5}$
   (d) $\frac{4}{10}$
   (e) $\frac{3}{1}$
2. Find a solution in $(Q, \cdot)$ for each of the following open sentences. Write each answer as an irreducible fraction.
   (a) $5 \cdot x = 4$
   (b) $-7 \cdot x = 1$
   (c) $16 \cdot x = -8$
   (d) $6 \cdot x = -1$
   (e) $5 \cdot x = 2$
   (f) $10 \cdot x = 4$
   (g) $-15 \cdot x = -6$
   (h) $25 \cdot x = 10$
3. For each of the following rational numbers, write an equation for which it is a solution.

(a) $\frac{7}{9}$ (d) $\frac{1}{-3}$

(b) $\frac{-12}{13}$ (e) $\frac{-1}{3}$

(c) $\frac{1}{2}$ (f) $\frac{-5}{-8}$

4. Determine which of the following statements are true. Then use procedures similar to those of Section 12.4 to check your answer.

(a) $\frac{2}{3} = \frac{6}{9}$

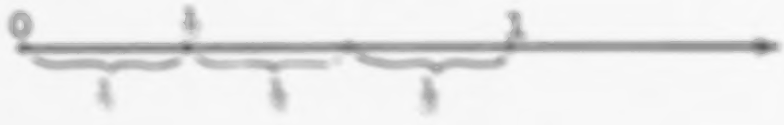
(b) $\frac{5}{8} = \frac{-15}{-24}$

(c) $\frac{47}{94} = \frac{-2}{4}$

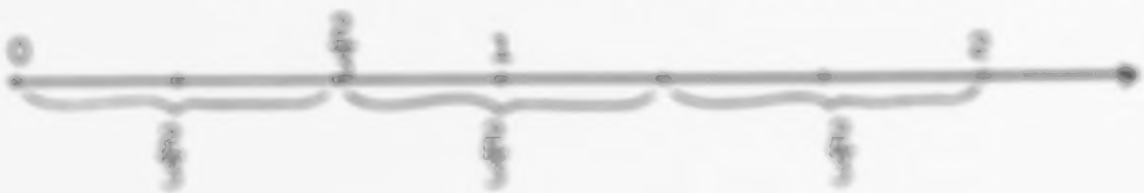
(d) $\frac{2}{-7} = \frac{-4}{14}$ (Hint: $\frac{1}{-1} = -1 = \frac{-1}{1}$)

(e) $\frac{4}{5} = \frac{19}{20}$

5. Rational numbers, as well as whole numbers and integers, can be represented on a number line. For instance, $\frac{1}{3}$ is located as indicated.



because $3 \cdot \frac{1}{3} = 1$. Similarly, $\frac{2}{3}$ is located as indicated



because $3 \cdot \frac{2}{3} = 2$

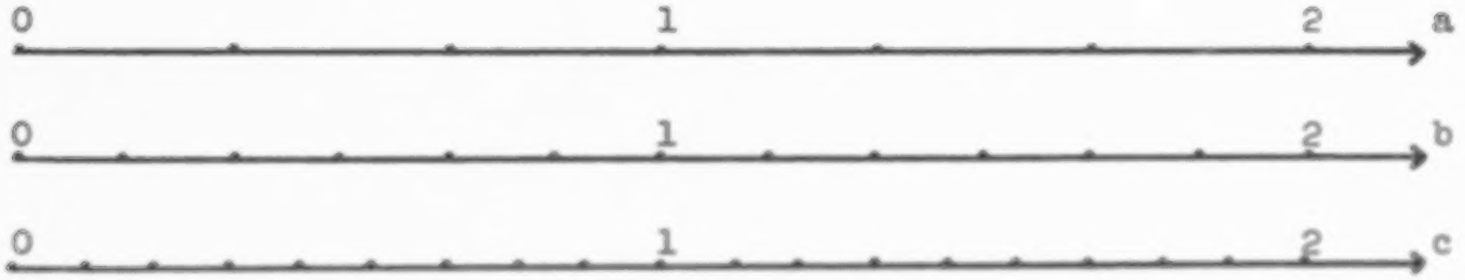
(a) Draw similar number lines to illustrate the locations of:

(1) $\frac{1}{2}$ (4) $\frac{3}{4}$

(2) $\frac{1}{4}$ (5) $\frac{3}{8}$

(3) $\frac{1}{8}$ (6) $\frac{2}{8}$

(b) Draw three parallel number lines and scale them by tracing the scales on the following lines.



Locate $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{6}{9}$ on lines a, b, and c respectively and show why each is a solution of "$3 \cdot x = 2$."

12.6 (Q,·)

Is the new system (Q,·) an operational system? If p and q are any elements of Q, is there a unique element r of Q assigned as the product of p and q? Remember Q consists of the integers, the multiplicative inverses of all integers except 0, and the products $a \cdot \frac{1}{b}$, where a and b are integers and $b \neq 0$.

If p and q are both integers, for example 17 and -12, the product $(17)(-12) = -204$ is an integer and in Q. If p and q are inverses of integers, for example $\frac{1}{-7}$ and $\frac{1}{3}$, the product

$(\frac{1}{-7})(\frac{1}{3}) = \frac{1}{-21}$ is the inverse of an integer and in Q. If p is an integer and q the inverse of an integer, for example -9 and $\frac{1}{6}$, the product is $(-9) \cdot (\frac{1}{6}) = \frac{-9}{6}$, again an element of Q.

But what if $p = \frac{7}{-8}$ and $q = \frac{2}{3}$? How is a product assigned to (p,q)? Is $p \cdot q$ an element of Q?

$$\begin{aligned}(\tfrac{7}{-8}) \cdot (\tfrac{2}{3}) &= [7 \cdot (\tfrac{1}{-8})] \cdot [2 \cdot \tfrac{1}{3}] \\ &= (7 \cdot 2)(\tfrac{1}{-8} \cdot \tfrac{1}{3}) \\ &= (14)(\tfrac{1}{-24}) \\ &= \tfrac{14}{-24}\end{aligned}$$

$\frac{14}{-24}$ is an element of Q. In general:

$$\begin{aligned}\tfrac{a}{b} \cdot \tfrac{c}{d} &= (a \cdot \tfrac{1}{b})(c \cdot \tfrac{1}{d}) && (1) \\ &= (a \cdot c)(\tfrac{1}{b} \cdot \tfrac{1}{d}) && (2) \\ &= (a \cdot c)(\tfrac{1}{b \cdot d}) && (3) \\ &= \tfrac{a \cdot c}{b \cdot d} && (4)\end{aligned}$$

It is important to recognize several assumptions that allow deduction of the computational rule $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$. (1) is true by definition -- $\frac{a}{b} = a \cdot \frac{1}{b}$ and $\frac{c}{d} = c \cdot \frac{1}{d}$. The derivation of (2) from (1) is made possible by the assumption that $\cdot$ is both commutative and associative.

$$(a \cdot \frac{1}{b})(c \cdot \frac{1}{d}) = [a \cdot (\frac{1}{b} \cdot c)] \cdot \frac{1}{d} \qquad \text{Assoc.}$$

$$= [a \cdot (c \cdot \frac{1}{b})] \cdot \frac{1}{d} \qquad \text{Comm.}$$

$$= (a \cdot c)(\frac{1}{b} \cdot \frac{1}{d}) \qquad \text{Assoc.}$$

(3) follows logically from (2) since $\frac{1}{b} \cdot \frac{1}{d} = \frac{1}{b \cdot d}$ (See Section 12.2). (4) follows from (3) because $x \cdot \frac{1}{y} = \frac{x}{y}$ for any integers $\underline{x}$ and $\underline{y}$. Thus the rules for computing products in Q are a direct consequence of our definitions, of multiplication in Z, and of the properties of (Z,·) that we want to hold in (Q,·) as well.

Study the following examples of computations in (Q,·).

(1) $\frac{7}{8} \cdot \frac{1}{2} = \frac{7 \cdot 1}{8 \cdot 2} = \frac{7}{16}$

(2) $\frac{-8}{3} \cdot \frac{5}{-6} = \frac{(-8) \cdot 5}{3 \cdot (-6)} = \frac{-40}{-18}$

(3) $\frac{18}{19} \cdot \frac{-3}{2} = \frac{18 \cdot (-3)}{19 \cdot 2} = \frac{-54}{38}$

Question: What are the irreducible fraction names

for $\frac{7}{16}, \frac{-40}{-18}, \frac{-54}{38}$?

The next important question: Does each equation of the form "$a \cdot x = b$" have a solution in Q ($a, b \in Z, a \neq 0$)? Study the following examples.

Equation	Solution Set
$3 \cdot x = 2$	$\{\frac{2}{3}\}$
$-7 \cdot x = 5$	$\{\frac{5}{-7}\}$
$-11 \cdot x = -13$	$\{\frac{-13}{-11}\}$

These examples, and your previous experience with equations, should convince you that any equation "$a \cdot x = b$" (where $a \neq 0$) has a solution, $\frac{b}{a}$, in Q. Furthermore, every rational number is the solution of some such equation. Therefore, a rational number can be described as the solution of equation "$a \cdot x = b$," where a and b are integers and $a \neq 0$.

The following exercises give practice computing in $(Q, \cdot)$, and suggest important properties of this operational system. The question of whether division is an operation in $(Q, \cdot)$ is taken up in Section 12.10.

12.7 Exercises

1. Compute in $(Q, \cdot)$. Give your answer as an irreducible fraction.

   (a) $\frac{2}{3} \cdot \frac{1}{2}$

   (b) $\frac{7}{9} \cdot \frac{-3}{5}$

   (c) $\frac{5}{2} \cdot \frac{2}{2}$

   (d) $\frac{-5}{2} \cdot \frac{13}{-17}$

   (e) $\frac{3}{8} \cdot \frac{-5}{8}$

   (f) $\frac{5}{1} \cdot \frac{1}{5}$

   (g) $\frac{0}{5} \cdot \frac{-3}{11}$

   (h) $\frac{-5}{-8} \cdot \frac{1}{4}$

2. We know that $\frac{3}{4} = \frac{9}{12}$. Is the product $\frac{3}{4} \cdot \frac{2}{3} = \frac{9}{12} \cdot \frac{2}{3}$?

   It should be, since "$\frac{3}{4}$" and "$\frac{9}{12}$" are only different names for the same rational number.

   (a) Compute $\frac{3}{4} \cdot \frac{2}{3}$ and $\frac{9}{12} \cdot \frac{2}{3}$ and express your answers as irreducible fractions.

(b) Do your answers in (a) agree?

(c) Compute and express as irreducible fractions.

(1) $\frac{5}{8} \cdot \frac{1}{3}$ and $\frac{10}{16} \cdot \frac{1}{3}$

(2) $\frac{-2}{3} \cdot \frac{1}{7}$ and $\frac{4}{-6} \cdot \frac{1}{7}$

(3) $\frac{-152}{-16} \cdot \frac{32}{56}$ and $\frac{19}{2} \cdot \frac{4}{7}$

(4) $\frac{5}{6} \cdot \frac{6}{16}$ and $\frac{10}{12} \cdot \frac{3}{8}$

(5) $\frac{2}{3} \cdot \frac{1}{4}$ and $\frac{4}{6} \cdot \frac{2}{8}$

(d) Do your results in (c) confirm or deny the statement:

if $\frac{a}{b} = \frac{c}{d}$ and $\frac{a'}{b'} = \frac{c'}{d'}$ then $\frac{a \cdot a'}{b \cdot b'} = \frac{c \cdot c'}{d \cdot d'}$?

3. Find the following products of rational numbers.

(a) $\frac{3}{5} \cdot \frac{2}{9}$

(b) $\frac{5}{8} \cdot \frac{3}{7}$

(c) $\frac{10}{11} \cdot \frac{4}{5}$

(d) $\frac{4}{5} \cdot \frac{10}{11}$

(e) $\frac{10}{3} \cdot \frac{3}{1}$

(f) $\frac{7}{2} \cdot \frac{3}{5}$

(g) $\frac{3}{5} \cdot \frac{7}{2}$

(h) $\frac{4}{9} \cdot \frac{2}{7}$

(i) $\frac{2}{7} \cdot \frac{4}{9}$

(j) $\frac{0}{3} \cdot \frac{5}{8}$

(k) $(\frac{2}{3} \cdot \frac{4}{5}) \cdot \frac{7}{6}$

(l) $\frac{2}{3} \cdot (\frac{4}{5} \cdot \frac{7}{6})$

(m) $\frac{0}{3} \cdot (\frac{6}{6} \cdot \frac{3}{2})$

(n) $\frac{5}{8} \cdot (\frac{21}{2} \cdot \frac{4}{9})$

(o) $(\frac{5}{8} \cdot \frac{21}{2}) \cdot \frac{4}{9}$

4. Find each of the following products.

(a) $5 \cdot 2$

(b) $\frac{5}{1} \cdot \frac{2}{1}$

(c) $3 \cdot 6$

(d) $\frac{3}{1} \cdot \frac{6}{1}$

(e) $7 \cdot 8$

(f) $\frac{7}{1} \cdot \frac{8}{1}$

(g) $15 \cdot 5$

(h) $\frac{15}{1} \cdot \frac{5}{1}$

(i) $-2 \cdot 3$

(j) $\frac{-2}{1} \cdot \frac{3}{1}$

(k) $(-4)(-6)$

(l) $\frac{-4}{1} \cdot \frac{-6}{1}$

5. Determine the following products of rational numbers.

(a) $\frac{3}{3} \cdot \frac{2}{5}$ (d) $\frac{10}{7} \cdot \frac{8}{8}$

(b) $\frac{4}{3} \cdot \frac{7}{7}$ (e) $\frac{1000}{1000} \cdot \frac{9}{5}$

(c) $\frac{5}{6} \cdot \frac{1}{1}$ (f) $\frac{6}{6} \cdot \frac{3}{4}$

6. Determine the following products. Use an irreducible fraction to represent each product.

(a) $\frac{2}{3} \cdot \frac{3}{2}$ (e) $\frac{100}{7} \cdot \frac{7}{100}$

(b) $\frac{5}{7} \cdot \frac{7}{5}$ (f) $\frac{1}{1} \cdot \frac{1}{1}$

(c) $\frac{9}{4} \cdot \frac{4}{9}$ (g) $\frac{22}{5} \cdot \frac{5}{22}$

(d) $\frac{10}{3} \cdot \frac{3}{10}$ (h) $\frac{14}{99} \cdot \frac{99}{14}$

7. Below are a number of equations, each of which has a solution which is a rational number. For each equation, write the irreducible fraction which represents the solution. Then write four other fractions for the number.

(a) $7 \cdot x = 5$ (e) $5 \cdot x = 2$

(b) $15 \cdot x = 10$ (f) $10 \cdot x = 4$

(c) $4 \cdot x = 1$ (g) $-3 \cdot x = 2$

(d) $10 \cdot x = 1$ (h) $3 \cdot x = -2$

(i) $b \cdot x = a \; (b \neq 0)$

8. Is it true that $\frac{8}{11} = \frac{104}{143}$? If you are patient you will discover that 104 and 143 have 13 as a common factor. Thus $\frac{104}{143} = \frac{8 \cdot 13}{11 \cdot 13} = \frac{8}{11}$. It often easier to check such statements using the following rule:

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc.$$

(a) Check this rule with the known statements:

(1) $\frac{2}{3} = \frac{4}{6}$

(2) $\frac{-3}{10} = \frac{6}{-20}$

(3) $\frac{7}{8} = \frac{-14}{-16}$

(b) Using this rule determine which of the following statements are true.

(1) $\frac{20}{2} = \frac{100}{10}$

(2) $\frac{-15}{3} = \frac{10}{-2}$

(3) $\frac{6}{6} = \frac{-19}{-19}$

(4) $\frac{0}{5} = \frac{0}{9}$

(5) $\frac{45}{92} = \frac{32}{65}$

(6) $\frac{-8}{4} = \frac{10}{-5}$

(7) $\frac{-5}{8} = \frac{5}{-8}$

(8) $\frac{5}{6} = \frac{55}{65}$

(9) $\frac{2}{2} = \frac{1}{1}$

(10) $\frac{7}{7} = \frac{1}{1}$

12.8 Properties of $(Q, \cdot)$

As with all operational systems, it is worthwhile to investigate the properties of $(Q, \cdot)$. As was illustrated in Exercise 3 of Section 12.7, multiplication of rational numbers is both commutative and associative.

Commutative Property of $(Q, \cdot)$

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}.$$

Associative Property of $(Q, \cdot)$

If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are rational numbers, then

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right).$$

If you refer to Exercise 5 of Section 12.7, you should see that there is an identity element in $(Q, \cdot)$. This identity element is the rational number 1, also named by the following fractions: "$\frac{1}{1}$," "$\frac{-1}{-1}$," "$\frac{2}{2}$," "$\frac{-2}{-2}$," "$\frac{3}{3}$," ... .

Example 1: $\frac{3}{4} \cdot \frac{2}{2} = \frac{6}{8}$

$= \frac{3}{4}$

Example 2: $\frac{3}{4} \cdot \frac{5}{5} = \frac{15}{20}$

$= \frac{3}{4}$

Examples 1 and 2 are really the same rational number products. In both cases, the rational number $\frac{3}{4}$ was multiplied by the same rational number; the only difference is that in the first example the fraction "$\frac{2}{2}$" was used to represent the number, while in the second example the fraction "$\frac{5}{5}$" was used. In both cases the product was $\frac{3}{4}$ since the fractions "$\frac{2}{2}$" and "$\frac{5}{5}$" represent the identity element, 1, of $(Q, \cdot)$.

Identity Element of $(Q, \cdot)$

If $\frac{a}{b}$ is a rational number, then $\frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b}$, that is,

$$\frac{a}{b} \cdot 1 = \frac{a}{b}.$$

What is the product of $\frac{2}{3}$ and $\frac{3}{2}$? It is easy to check that the product is $\frac{6}{6}$ or 1, the identity element of $(Q, \cdot)$; therefore, these rational numbers are inverses of each other in this system. If you refer to Exercise 6 of Section 12.7 you should notice a pattern -- the inverse of $\frac{a}{b}$ is $\frac{b}{a}$. There is one important exception to this rule however. The product of $\frac{0}{1}$ and another rational number cannot be 1.

Note that $\frac{0}{2}$ names the same rational number as $\frac{0}{1}$ as does $\frac{0}{n}$ for any non-zero integer *n*.

Question: If $\frac{a}{b}$ is any rational number, what is the product $\frac{0}{1} \cdot \frac{a}{b}$? Do you see then why $\frac{0}{1}$ has no inverse in $(Q,\cdot)$?

We now state the following property:

Inverse Property of $(Q,\cdot)$

If $\frac{a}{b}$ is a rational number which is not $\frac{0}{1}$ (that is, $a \neq 0$), then $\frac{b}{a}$ is the inverse of $\frac{a}{b}$, that is $\frac{a}{b} \cdot \frac{b}{a} = 1$.

Thus, the rational numbers $\frac{a}{b}$ and $\frac{b}{a}$ are inverses in $(Q,\cdot)$.

Because the operation in this system is multiplication, we may call them *multiplicative inverses*. It is also common in the system $(Q,\cdot)$ to call a multiplicative inverse a *reciprocal*.

Example 3: The multiplicative inverse of the number $\frac{2}{3}$ is $\frac{3}{2}$ or the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Each of 5, $\frac{1}{5}$ is the reciprocal of the other.

12.9 Exercises

1. For each of the following equations, find a solution in $(Q,\cdot)$.

   (a) $\frac{2}{3} \cdot a = \frac{2}{3}$

   (b) $\frac{4}{3} \cdot a = 1$

   (c) $\frac{10}{9} \cdot \frac{9}{10} = x$

   (d) $\frac{3}{3} \cdot x = \frac{5}{6}$

   (e) $\frac{4}{5} \cdot m = \frac{1}{1}$

   (f) $\frac{10}{7} \cdot a = 1$

   (g) $\frac{15}{4} \cdot x = \frac{15}{4}$

   (h) $x \cdot x = 1$

2. Determine each of the following products:

   (a) $\frac{0}{5} \cdot \frac{2}{3}$

   (b) $\frac{5}{9} \cdot 0$

   (c) $\frac{23}{11} \cdot \frac{0}{1}$

   (d) $0 \cdot \frac{4}{5}$

   (e) $\frac{0}{17} \cdot \frac{35}{8}$

   (f) $\frac{0}{1} \cdot 1$

3. The rational number 0 is represented by any one of the fractions in the set

$$\{\ldots, \text{"}\tfrac{0}{-2}\text{,"}\ \text{"}\tfrac{0}{-1}\text{,"}\ \text{"}\tfrac{0}{1}\text{,"}\ \text{"}\tfrac{0}{2}\text{,"}\ \text{"}\tfrac{0}{3}\text{,"}\ \ldots\}.$$

   On the basis of the products in Problem 2, how would you describe the behavior of this number in multiplication?

4. (a) List 10 fractions which name the identity element in $(Q,\cdot)$.

   (b) List 10 fractions which name the inverse of $\frac{2}{5}$ in $(Q,\cdot)$.

   (c) What is the product of $\frac{3}{4}$ and $\frac{8}{6}$?

   (d) What rational number is its own inverse in the system $(Q,\cdot)$?

(e) What rational number has no inverse in the system $(Q,\cdot)$?

5. (a) Write the properties which a system $(S,*)$ must have in order to be a group. (See Chapter 2, Section 2.15.)

(b) Is $(Z,\cdot)$ a group? If so, is it commutative?

(c) Is $(Q,\cdot)$ a group? If so, it it commutative?

(d) Let X be the set of all rational numbers except 0. Is $(X,\cdot)$ a group? If so, is it commutative?

6. (a) Compute the following products in $(Z,\cdot)$:

$-8\cdot 1$, $14\cdot 1$, $-234\cdot 1$, $55\cdot 1$, $86\cdot 0$, $-14\cdot 0$.

(b) Compute the following products in $(Q,\cdot)$:

$\frac{-8}{1}\cdot\frac{1}{1}$, $\frac{14}{1}\cdot\frac{1}{1}$, $\frac{-234}{1}\cdot\frac{1}{1}$, $\frac{55}{1}\cdot\frac{1}{1}$, $\frac{86}{1}\cdot\frac{0}{1}$, $\frac{-14}{1}\cdot\frac{0}{1}$

7. Often a short cut can be used in finding the product of two rational numbers. Perhaps you have used this short cut before, but have never been able to explain why it works.

Study the following example:

$$\frac{2}{3}\cdot\frac{5}{6}=\frac{2\cdot 5}{3\cdot 6}=\frac{2\cdot 5}{3\cdot(2\cdot 3)}=\frac{2\cdot 5}{2\cdot(3\cdot 3)}=\frac{2}{2}\cdot\frac{5}{3\cdot 3}=\frac{5}{9}$$

This is not a short cut! But notice that since $\frac{2}{2}$ is the identity element for multiplication, we could have determined the product this way:

$$\frac{\overset{1}{\cancel{2}}}{3}\cdot\frac{5}{\underset{3}{\cancel{6}}}=\frac{5}{9}$$

Do you see how the identity element for multiplication has been used in the following example?

$$\frac{\overset{2}{\cancel{4}}}{{}_{5}\cancel{15}} \cdot \frac{\overset{3}{\cancel{9}}}{\underset{5}{\cancel{10}}} = \frac{6}{25}$$

Use this short cut in finding the following products:

(a) $\frac{7}{3} \cdot \frac{5}{14}$

(b) $\frac{5}{8} \cdot \frac{7}{5}$

(c) $\frac{10}{9} \cdot \frac{27}{2}$

(d) $\frac{5}{2} \cdot \frac{2}{5}$

(e) $\frac{18}{45} \cdot \frac{15}{27}$

(f) $\frac{5}{4} \cdot (\frac{24}{7} \cdot \frac{7}{36})$

(g) $(\frac{3}{3} \cdot \frac{9}{10}) \cdot \frac{15}{27}$

(h) $(\frac{24}{11} \cdot \frac{33}{42}) \cdot \frac{1}{2}$

(i) $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7}$

(j) $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \cdot \frac{d}{e} \cdot \frac{e}{f}$

12.10 Division of Rational Numbers

In $(Z, \cdot)$, the equation

$$12 \div 3 = x$$

has the solution 4, because $4 \cdot 3 = 12$. That is, division is defined in terms of multiplication. We define division in this way also in $(Q, \cdot)$.

Definition: If a and b are rational numbers then

$$b \div a = x \text{ if and only if } x \cdot a = b.$$

Suppose we want to find $\frac{3}{4} \div \frac{2}{5}$. We must then find a rational number $\frac{x}{y}$ such that

$$\frac{3}{4} \div \frac{2}{5} = \frac{x}{y}$$

Is there a solution? If there is, we want the following to be true:

$$\frac{x}{y} \cdot \frac{2}{5} = \frac{3}{4}.$$

Now the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$, and we know that

$$\frac{5}{2} \cdot \frac{2}{5} = \frac{1}{1} .$$

Therefore,

$$\frac{3}{4} \cdot (\frac{5}{2} \cdot \frac{2}{5}) = \frac{3}{4} .$$

Using the associative property of multiplication, we can write

$$(\frac{3}{4} \cdot \frac{5}{2}) \cdot \frac{2}{5} = \frac{3}{4} .$$

Do you see that we have found the number $\frac{x}{y}$ which we were trying to find? It is the product $\frac{3}{4} \cdot \frac{5}{2}$, which is the rational number $\frac{15}{8}$.

So, $\frac{3}{4} \cdot \frac{5}{2}$ is the solution of $\frac{3}{4} \div \frac{2}{5} = \frac{x}{y}$. In other words,

$$\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \cdot \frac{5}{2} .$$

From this example, it would seem that the quotient of two rational numbers can be found by finding the product of two rational numbers.

See if you can follow the steps in the following example:

$$\frac{4}{3} \div \frac{3}{2} = \frac{x}{y}$$

$$\frac{x}{y} \cdot \frac{3}{2} = \frac{4}{3}$$

Now,

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{1}{1}$$

So,

$$\frac{4}{3} \cdot (\frac{2}{3} \cdot \frac{3}{2}) = \frac{4}{3}$$

$$(\frac{4}{3} \cdot \frac{2}{3}) \cdot \frac{3}{2} = \frac{4}{3} .$$

So we have found the rational number whose product with $\frac{3}{2}$ is $\frac{4}{3}$ and that is the number $\frac{x}{y}$ which we were seeking. Therefore,

$$\frac{4}{3} \div \frac{3}{2} = \frac{4}{3} \cdot \frac{2}{3} \qquad \text{(which of course is } \frac{8}{9}\text{)}.$$

Thus, to divide by $\frac{3}{2}$, you multiply by the reciprocal of $\frac{3}{2}$. And if you look at the first example again, you see the same pattern there: to divide by $\frac{2}{5}$, you multiply by the reciprocal of $\frac{2}{5}$.

Finally, let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers ($c \neq 0$).

If $\frac{a}{b} \div \frac{c}{d} = \frac{x}{y}$, then $\frac{x}{y} \cdot \frac{c}{d} = \frac{a}{b}$.

But we know $(\frac{a}{b} \cdot \frac{d}{c}) \cdot \frac{c}{d} = \frac{a}{b}$. (Why? Can you supply the missing step?)

So $\frac{x}{y} = \frac{a}{b} \cdot \frac{d}{c}$. That is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Can you complete the following sentence?

Dividing by the rational number $\frac{x}{y}$ is equivalent to multiplying ________________________.

You will recall that in Section 12.6, a rational number $\frac{b}{a}$ was described as the solution of an equation $ax = b$ where $a \in Z$, $b \in Z$ and $a \neq 0$. Now let us consider an equation of this type where a and b are in Q, and $a \neq 0$. One might be tempted to think that in this case some kind of new number might be called for as a solution. But this is not the case. Consider the equation

$$\frac{2}{3} \cdot x = \frac{5}{6}$$

If you examine the definition of division carefully you will see that the rational number x must be the quotient of $\frac{5}{6}$ and $\frac{2}{3}$. That is,

$$x = \frac{5}{6} \div \frac{2}{3}$$

Hence, $x = \frac{5}{6} \cdot \frac{3}{2} = \frac{15}{12} = \frac{5}{4}$.

Checking this result, $\frac{2}{3} \cdot \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$.
Thus the rational number $\frac{5}{4}$ is a solution to this equation. It is a consequence of the properties of $(Q, +, \cdot)$ and the definition of division in $(Q, \cdot)$ that every equation "$\frac{a}{b} \cdot x = \frac{c}{d}$" with $\frac{a}{b} \in Q$, $\frac{c}{d} \in Q$ and $\frac{c}{d} \neq 0$ has the solution $\frac{c}{d} \cdot \frac{b}{a}$ in Q.

These results now make it possible for us to interpret any rational number $\frac{b}{a}$ not only as a solution of an equation $ax = b$ where $a \in Z$, $b \in Z$ and $a \neq 0$, but also as the quotient of two integers. That is, as $b \div a$.

In fact, the use of the capital letter "Q" to denote the rational numbers comes from the fact that a rational number is a quotient of integers.

Consider the rational number $\frac{5}{6}$. This is the solution of the equation "$6x = 5$." But by what we have said above, since 6 and 5 are also in Q, $5 \div 6$ is also a solution for this equation. That is, $\frac{5}{6} = 5 \div 6$.

Several consequences of this relationship are illustrated in the following examples.

Example 1. $\frac{0}{1} = 0 \div 1 = 0$, since $0 \cdot 1 = 0$.

In general, if $b \in Z$ and $b \neq 0$, $\frac{0}{b} = 0 \div b = 0$, since $0 \cdot b = 0$.

Example 2. $\frac{-6}{1} = -6 \div 1 = -6$, since $(-6) \cdot 1 = -6$.

In general, if $a \in Z$, $\frac{a}{1} = a \div 1 = a$, since $a \cdot 1 = a$.

Example 3. $\frac{-13}{-13} = -13 \div -13 = 1$ since $1 \cdot (-13) = -13$.

In general, if $a \in Z$ and $a \neq 0$, $\frac{a}{a} = a \div a = 1$, since $1 \cdot a = a$.

12.11 Exercises

1. Find the following quotients of rational numbers. Then use a product to show that your result is correct.

(a) $\frac{3}{8} \div \frac{1}{2}$

(b) $\frac{1}{2} \div \frac{3}{8}$

(c) $\frac{5}{4} \div \frac{2}{3}$

(d) $\frac{2}{3} \div \frac{5}{4}$

(e) $\frac{-7}{10} \div \frac{1}{12}$

(f) $\frac{1}{12} \div \frac{-7}{10}$

2. Find the following quotients of rational numbers.

(a) $\frac{14}{5} \div \frac{3}{7}$

(b) $\frac{8}{9} \div \frac{9}{8}$

(c) $\frac{8}{9} \div \frac{8}{9}$

(d) $\frac{0}{3} \div \frac{2}{5}$

(e) $\frac{4}{11} \div \frac{11}{4}$

(f) $\frac{4}{9} \div \frac{1}{2}$

(g) $\frac{4}{9} \div \frac{1}{3}$

(h) $\frac{4}{9} \div \frac{1}{4}$

(i) $(\frac{5}{4} \div \frac{1}{2}) \div \frac{2}{3}$

(j) $\frac{5}{4} \div (\frac{1}{2} \div \frac{2}{3})$

(k) $\frac{7}{3} \div (\frac{1}{3} \div \frac{2}{1})$

(l) $(\frac{7}{3} \div \frac{1}{3}) \div \frac{2}{1}$

3. Find the following quotients:

(a) $6 \div 2$

(b) $\frac{6}{1} \div \frac{2}{1}$

(c) $\frac{12}{2} \div \frac{8}{4}$

(d) $20 \div 5$

(e) $\frac{20}{1} \div \frac{5}{1}$

(f) $5 \div 5$

(g) $\frac{5}{1} \div \frac{5}{1}$

(h) $4 \div 8$

(i) $\frac{4}{1} \div \frac{8}{1}$

4. Determine a rational number solution of each of the following equations.

(a) $\frac{2}{3} \cdot \frac{x}{y} = \frac{3}{4}$

(b) $\frac{3}{4} \cdot \frac{x}{y} = \frac{2}{3}$

(c) $\frac{x}{y} \cdot \frac{2}{3} = \frac{5}{7}$

(d) $\frac{5}{7} \cdot \frac{2}{3} = \frac{x}{y}$

(e) $\frac{5}{6} \cdot \frac{x}{y} = \frac{4}{5}$

(f) $\frac{2}{3} \cdot \frac{x}{y} = \frac{4}{5}$

(g) $\frac{4}{9} \cdot \frac{x}{y} = \frac{2}{3}$

(h) $\frac{x}{y} \cdot \frac{2}{3} = \frac{7}{12}$

(i) $\frac{x}{y} \cdot \frac{14}{27} = \frac{0}{1}$

(j) $\frac{3}{2} \cdot \frac{x}{y} = \frac{1}{1}$

5. (a) Is it possible to find the quotient $\frac{2}{3} \div \frac{0}{1}$? Explain why or why not.

(b) What rational number has no reciprocal?

(c) In the sentence $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$,

what number must $\frac{c}{d}$ not be? Why?

(d) Is division an operation on the rational numbers? Why or why not?

(e) If the number 0 is removed from the set Q of rational numbers, is division an operation on the set of numbers that remain?

(f) Is division associative? (See Exercise 2.)

12.12 Addition of Rational Numbers

Since the rational numbers are an extension of the integers, we already know how to compute many sums in Q. For example, $-5 + 7 = 2$ or $\frac{-5}{1} + \frac{7}{1} = \frac{2}{1}$. Of course, it should not make any difference which of the many available fractions are used to represent the rational numbers $\frac{-5}{1}$ and $\frac{7}{1}$. This suggests the following:

$$\frac{-15}{3} + \frac{21}{3} = \frac{6}{3}\ ;\ \frac{-10}{2} + \frac{14}{2} = \frac{4}{2}$$

This in turn suggests that we define addition of rational numbers in the following way:

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

That is, in determining a sum we select fractions which have the same denominator.

Example 1: What is the sum of the rational numbers

$\frac{5}{3}$ and $\frac{2}{3}$?

$$\frac{5}{3} + \frac{2}{3} = \frac{5 + 2}{3} = \frac{7}{3}$$

This definition of addition in Q was suggested by the desire to extend addition of integers. There is another reason for adopting the above definition.

$$\frac{3}{5} + \frac{4}{5} = 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5}$$

If the distributive property of multiplication over addition is to hold in (Q,+,·):

$$3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5} = (3 + 4) \cdot \frac{1}{5}$$
$$= 7 \cdot \frac{1}{5}$$
$$= \frac{7}{5}$$

Example 2: What is the sum of the rational numbers $\frac{2}{3}$ and $\frac{3}{4}$? We may indicate the sum this way:

$$\frac{2}{3} + \frac{3}{4} .$$

However, in order to use the method above, we must find other fractions for these numbers, fractions with the same denominator. Now, the least common multiple of 3 and 4 is 12. So we say that 12 is the least common denominator of the denominators 3 and 4. We then represent each of the rational numbers by a fraction with denominator 12.

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$$

Although we do not prove it here, it is true that there is one and only one rational number which is the sum of two given rational numbers. For instance, in Example 2, we could have used the fractions $\frac{16}{24}$ and $\frac{18}{24}$. (Why?) Then the sum would have been the number represented by the fraction "$\frac{34}{24}$." But this is the same as the number $\frac{17}{12}$. (Why?)

In order to get a general definition from the method we have been using, let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers. Then to find the sum $\frac{a}{b} + \frac{c}{d}$, we need to select two fractons that have the same denominator. do you see that

$$\frac{a}{b} = \frac{ad}{bd} \text{ and } \frac{c}{d} = \frac{bc}{bd} \text{ ?}$$

Thus, we have:

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd}\end{aligned}$$

We now have an operational system (Q,+). In this system there are the following properties:

Commutative Property of Addition

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$.

Associative Property of Addition

If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are rational numbers,

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) .$$

Although we do not prove these properties here, there are examples of each of them in the exercises.

Now consider the rational number 0, also named by

$$\text{"}\frac{0}{1}\text{", "}\frac{0}{2}\text{", "}\frac{0}{3}\text{", "}\frac{0}{4}\text{", } \ldots$$

What are the following sums:

$$\frac{2}{3} + \frac{0}{3},\ \frac{5}{6} + \frac{0}{6},\ \frac{-2}{7} + \frac{0}{7},\ \frac{3}{4} + \frac{0}{4} .$$

For any rational number $\frac{a}{b}$, $\frac{a}{b} + \frac{0}{b} = \frac{a + 0}{b} = \frac{a}{b}$. You recognize here the familiar pattern for an identity element; and since the fraction $\frac{0}{b}$ represents the rational number 0 we have the following property:

Identity Element for Addition

For any rational number $\frac{a}{b}$, $\frac{a}{b} + 0 = \frac{a}{b}$.

In investigating operational systems in the past, the notion of inverse has been tied closely to that of identity element. Two elements are inverses of each other if together they produce the identity element. In this connection, study the following examples:

$$\frac{3}{4} + \frac{-3}{4} = \frac{3 + (-3)}{4} = \frac{0}{4} \qquad \frac{-5}{6} + \frac{5}{6} = \frac{-5 + 5}{6} = \frac{0}{6} .$$

These and similar examples should make the following property clear:

Inverse Elements for Addition

If $\frac{a}{b}$ is a rational number, then $\frac{a}{b} + \frac{-a}{b} = \frac{0}{1}$.

(-a is the additive inverse of a in the set Z of integers.)

That is, every rational number $\frac{a}{b}$ has an inverse, $\frac{-a}{b}$.

Example 3: What is the inverse of $\frac{6}{5}$ in (Q,+)?

The inverse is $\frac{-6}{5}$, $\frac{6}{5} + \frac{-6}{5} = \frac{0}{1}$.

Example 4. What is the inverse of $\frac{-3}{4}$?

In Z, the additive inverse of -3 is 3; that is, - (-3) = 3. So the additive inverse of $\frac{-3}{4}$ in Q is $\frac{3}{4}$, $\frac{-3}{4} + \frac{3}{4} = \frac{0}{1}$.

So far we have considered only properties of addition and multiplication separately. In $(Z,+,\cdot)$ the distributive property of multiplication over addition holds. That is for a, b and c in Z, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. Study the following examples:

$$\frac{2}{3}\left(\frac{1}{5} + \frac{1}{7}\right) = \frac{2}{3}\left(\frac{7}{35} + \frac{5}{35}\right) = \frac{2}{3} \cdot \frac{12}{35} = \frac{24}{105}$$

$$\left(\frac{2}{3} \cdot \frac{1}{5}\right) + \left(\frac{2}{3} \cdot \frac{1}{7}\right) = \frac{2}{15} + \frac{2}{21} = \frac{14}{105} + \frac{10}{105} = \frac{24}{105}$$

$$\frac{-3}{5}\left(\frac{3}{10} + \frac{-2}{15}\right) = \frac{-3}{5}\left(\frac{9}{30} + \frac{-4}{30}\right) = \frac{-3}{5} \cdot \frac{5}{30} = \frac{-1}{10}$$

$$\left(\frac{-3}{5} \cdot \frac{3}{10}\right) + \left(\frac{-3}{5} \cdot \frac{-2}{15}\right) = \frac{-9}{50} + \frac{6}{75} = \frac{-9}{50} + \frac{4}{50} = \frac{-5}{50} = \frac{-1}{10}$$

It appears that the distributive property holds in $(Q,+,\cdot)$ also.

Distributive Property for Multiplication over Addition

If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are rational numbers, then

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) + \left(\frac{a}{b} \cdot \frac{e}{f}\right).$$

We will agree that $2 \cdot 3 + 2 \cdot 5 = 6 + 10$ and that $3 \cdot 4 + 7 = 12 + 7$. That is, we take $2 \cdot 3 + 2 \cdot 5$ to mean $(2 \cdot 3) + (2 \cdot 5)$ and $3 \cdot 4 + 7$ to mean $(3 \cdot 4) + 7$. With this agreement the distributive property can be written as

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}.$$

This convention is followed in any operational system where multiplication and addition are defined. Thus, for integers a, b, and c we write $a\,(b + c) = a \cdot b + a \cdot c$.

12.13 Exercises

1. Find the following sums of rational numbers.

(a) $\frac{1}{2} + \frac{1}{3}$

(b) $\frac{2}{3} + \frac{3}{2}$

(c) $\frac{5}{6} + \frac{-2}{6}$

(d) $\frac{10}{7} + -\frac{3}{2}$

(e) $\frac{14}{9} + \frac{5}{3}$

(f) $\frac{20}{9} + \frac{5}{12}$

(g) $\frac{20}{9} + \frac{-5}{12}$

(h) $\frac{-7}{12} + \frac{13}{16}$

(i) $\frac{3}{4} + \frac{5}{-6}$

(Hint: "$\frac{-5}{6}$" represents the same rational number as "$\frac{5}{-6}$".

(j) $\frac{x}{y} + \frac{w}{z}$

2. What rational number is assigned to each of the following ordered pairs by the operation of addition?

(a) $(\frac{2}{5}, \frac{3}{10})$

(b) $(\frac{3}{10}, \frac{2}{5})$

(c) $(\frac{7}{8}, \frac{-3}{20})$

(d) $(\frac{-3}{20}, \frac{7}{8})$

(e) $(\frac{5}{6}, \frac{3}{32})$

(f) $(\frac{3}{32}, \frac{5}{6})$

(g) $(\frac{-1}{13}, \frac{1}{17})$

(h) $(\frac{1}{17}, \frac{-1}{13})$

3. What property of $(Q,+)$ do the sums in Exercise 2 illustrate?

4. Compute the following:

(a) $(\frac{2}{5} + \frac{1}{3}) + \frac{3}{2}$

(b) $\frac{2}{5} + (\frac{1}{3} + \frac{3}{2})$

(c) $(-\frac{3}{4} + \frac{5}{6}) + \frac{3}{8}$

(d) $-\frac{3}{4} + (\frac{5}{6} + \frac{3}{8})$

5. What property of $(Q,+)$ do the sums in Exercise 4 illustrate?

6. List ten different fractions which represent the identity element in (Q,+).

7. Compute the following:

(a) $\frac{8}{3} + \frac{-8}{3}$

(b) $\frac{8}{3} + \frac{-16}{6}$

(c) $\frac{9}{11} + \frac{0}{1}$

(d) $\frac{9}{11} + \frac{0}{11}$

(e) $\frac{-3}{14} + \frac{3}{14}$

(f) $\frac{148}{3} + \frac{-148}{3}$

(g) $\frac{148}{3} + \frac{148}{-3}$

(h) $\frac{-81}{7} + \frac{0}{51}$

8. Compute the following sums:

(a) $7 + 3$

(b) $\frac{7}{1} + \frac{3}{1}$

(c) $\frac{14}{2} + \frac{2}{3}$

(d) $0 + 7$

(e) $\frac{0}{1} + \frac{7}{1}$

(f) $-15 + 7$

(g) $\frac{-15}{1} + \frac{-7}{1}$

(h) $-8 + (-4)$

(i) $\frac{-8}{1} + \frac{-4}{1}$

9. (a) Is (Z,+) a group? If so, is it commutative?

(b) Is (Q,+) a group? If so, is it commutative?

(See Section 12.9, Exercise 5.)

10. Give the additive inverse of each of the following rational numbers.

(a) $\frac{2}{3}$

(b) $\frac{-5}{3}$

(c) $\frac{0}{1}$

(d) $\frac{15}{7}$

(e) $\frac{-15}{7}$

(f) $\frac{-a}{b}$

11. If we use "$- \frac{a}{b}$" to denote the additive inverse of the rational number $\frac{a}{b}$, complete each of the following so as to have a true statement.

(a) $-\frac{3}{4} =$ (d) $-\frac{-75}{7} =$ (g) $-\frac{-7}{8} =$

(b) $-\frac{-5}{2} =$ (e) $-\frac{2}{5} =$ (h) $-(-\frac{-7}{8}) =$

(c) $-\frac{10}{3} =$ (f) $-(-\frac{2}{5}) =$ (i) $-(-\frac{a}{b}) =$

12. Compute the following:

(a) $\frac{1}{2} \cdot (\frac{1}{6} + \frac{7}{6})$

(b) $(\frac{1}{2} \cdot \frac{1}{6}) + (\frac{1}{2} \cdot \frac{7}{6})$

(c) $\frac{2}{3} \cdot (\frac{5}{8} + \frac{3}{8})$

(d) $(\frac{2}{3} \cdot \frac{5}{8}) + (\frac{2}{3} \cdot \frac{3}{8})$

(e) $(\frac{7}{5} + \frac{-3}{5}) \cdot \frac{3}{4}$

(f) $(\frac{7}{5} \cdot \frac{3}{4}) + (\frac{-3}{5} \cdot \frac{3}{4})$

(g) $\frac{2}{5} \cdot (\frac{1}{2} + \frac{2}{3})$

(h) $(\frac{2}{5} \cdot \frac{1}{2}) + (\frac{2}{5} \cdot \frac{2}{3})$

12.14 Subtraction of Rational Numbers

In (Z,+), we say

$$5 - 3 = 2, \text{ because } 2 + 3 = 5.$$

And, in general,

$$\text{if } c + b = a, \text{ thean } a - b = c.$$

In other words, subtraction is defined in terms of addition. We shall make the same sort of definition in (Q,+).

Definition: If a, b, and c are rational numbers, then

$$a - b = c \text{ if } c + b = a.$$

For example, since $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$, we agree that

$$\frac{3}{5} - \frac{1}{5} = \frac{2}{5} .$$

And $\frac{2}{5}$ is the difference between $\frac{3}{5}$ and $\frac{1}{5}$, or the result of subtracting $\frac{1}{5}$ from $\frac{3}{5}$. We could have found this difference in the following way:

$$\frac{2}{5} + \frac{^-1}{5} = \frac{1}{5} .$$

That is, instead of subtracting $\frac{1}{5}$, we might add the additive inverse of $\frac{1}{5}$. This is, of course, the same pattern we noticed earlier for the integers. We consider below the general case for the rational numbers.

$$\text{Let } \frac{a}{b} - \frac{c}{d} = \frac{x}{y}.$$

Then by definition of subtraction,

$$\frac{x}{y} + \frac{c}{d} = \frac{a}{b}$$

$$\left(\frac{x}{y} + \frac{c}{d}\right) + \frac{^-c}{d} = \frac{a}{b} + \frac{^-c}{d}$$

$$\frac{x}{y} + \left(\frac{c}{d} + \frac{^-c}{d}\right) = \frac{a}{b} + \frac{^-c}{d}$$

$$\frac{x}{y} + \frac{0}{1} = \frac{a}{b} + \frac{^-c}{d}$$

$$\frac{x}{y} = \frac{a}{b} + \frac{^-c}{d}$$

But in our original equation,

$$\frac{x}{y} = \frac{a}{b} - \frac{c}{d}.$$

Therefore,

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} .$$

As a practical matter then we can always find a sum instead of a difference, provided we remember to add the inverse of the number being subtracted.

Example: $\frac{-3}{5} - \frac{-2}{3} = \frac{-3}{5} + \frac{2}{3}$

$$= \frac{-9}{15} + \frac{10}{15}$$

$$= \frac{-9 + 10}{15}$$

$$= \frac{1}{15}$$

12.15 Exercises

1. Compute the following differences.

(a) $\frac{3}{5} - \frac{1}{5}$

(b) $\frac{10}{13} - \frac{5}{13}$

(c) $\frac{5}{13} - \frac{10}{13}$

(d) $\frac{2}{3} - \frac{11}{3}$

(e) $\frac{3}{5} - \frac{-1}{5}$

(f) $\frac{-3}{5} - \frac{1}{5}$

(g) $\frac{-3}{5} - \frac{-1}{5}$

(h) $\frac{7}{8} - \frac{3}{4}$

(i) $\frac{6}{8} - \frac{3}{4}$

(j) $\frac{5}{6} - \frac{21}{8}$

(k) $\frac{5}{6} - \frac{-21}{8}$

(l) $\frac{-5}{6} - \frac{21}{8}$

(m) $\frac{-5}{6} - \frac{-21}{8}$

(n) $\frac{2}{13} - \frac{3}{15}$

2. (a) What is the difference $\frac{2}{3} - \frac{3}{5}$?

(b) What is the sum $\frac{2}{3} + \frac{-3}{5}$?

(c) What number does "$\frac{3}{5}$" name?

(d) What is the sum $\frac{2}{5} + (- \frac{3}{5})$?

3. Compute the following:

(a) $\frac{7}{8} - \frac{3}{8}$

(b) $\frac{7}{8} - \frac{-3}{8}$

(c) $\frac{7}{8} + (- \frac{3}{8})$

(d) $\frac{5}{12} + (- \frac{13}{16})$

(e) $\frac{5}{3} + (- \frac{3}{7})$

(f) $\frac{5}{3} - \frac{3}{7}$

(g) $\frac{7}{12} + (- \frac{7}{12})$

(h) $\frac{7}{12} - \frac{7}{12}$

4. Is subtraction a binary operation on the set Q of rational numbers?

5. (a) Is subtraction of rational numbers associative?

   (b) Is subtraction of rational numbers commutative?

   (c) Is there an identity element in (Q,-)?

6. Is (Q,-) a group? Why or why not? (See Section 12.9 Exercise 5).

12.16 Ordering the Rational Numbers

In the set Z of integers we know that $2 < 3$. Since 2 and 3 are also elements of Q, this suggests that the order relation "<" be defined in Q so that it is still true that $2 < 3$. Also, $-7 < -4$ and $-1 < 5$. We could write these in Q as

$$\frac{2}{1} < \frac{3}{1}, \quad \frac{-7}{1} < \frac{-4}{1}, \quad \text{and } \frac{-1}{1} < \frac{5}{1},$$

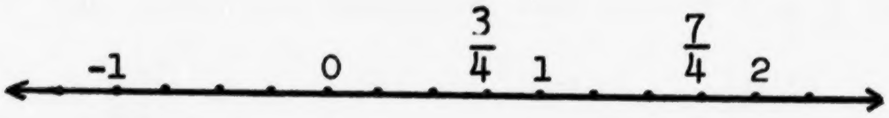
or as

$$\frac{4}{2} < \frac{6}{2}\,, \quad \frac{-21}{3} < \frac{-12}{3}\,, \quad \text{and } \frac{-4}{4} < \frac{20}{4}\,.$$

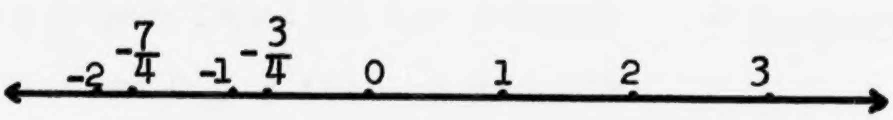
These examples further suggest that we agree to the following:

In Q, $\frac{a}{b} < \frac{c}{b}$ if and only if $b > 0$ and $a < c$ in Z.

Example 1: $\frac{3}{4} < \frac{7}{4}$ in Q, since $4 > 0$ and $3 < 7$ in Z. Notice that if we represent the rational numbers $\frac{3}{4}$ and $\frac{7}{4}$ on a number line, the point whose coordinate is $\frac{3}{4}$ is to the left of the point whose coordinate is $\frac{7}{4}$.



Example 2: $\frac{-7}{4} < \frac{-3}{4}$ since $4 > 0$ and $-7 < -3$ in Z. Again, if we represent the rational numbers $\frac{-7}{4}$ and $\frac{-3}{4}$ on a number line, the point whose coordinate is $\frac{-7}{4}$ is to the left of the point whose coordinate is $\frac{-3}{4}$.



Example 3: Compare the rational numbers $\frac{11}{13}$ and $\frac{7}{9}$. Which is less? Our method for comparing rational numbers is based on fractions that have the same denominator. Therefore, we shall use

the fractions

$$\frac{11 \cdot 9}{13 \quad 9} \text{ and } \frac{7 \cdot 13}{9 \cdot 13}$$

to compare the given rational numbers.

(Do you see why these fractions were chosen?)

Now, since $13 \cdot 9 > 0$ and $7 \cdot 13 < 11 \cdot 9$ in Z.

we have

$$\frac{7}{9} < \frac{11}{13} \text{ in Q.}$$

From Example 3, we notice that $\frac{7}{9} < \frac{11}{13}$ since $7 \cdot 13 < 9 \cdot 11$. And this suggests a general way of comparing two rational numbers without actually writing fractions with the same denominator. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, and b and d are both positive integers. Then the fractions "$\frac{ad}{bd}$" and "$\frac{bc}{bd}$" also represent these numbers. (Why?) And by our earlier agreement,

$$\frac{ad}{bd} < \frac{bc}{bd} \text{ if and only if } ad < bc.$$

Therefore, we make the following definition for ordering rational numbers:

Definition: If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, and b and d are positive integers, $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$.

Example 4: Compare the rational numbers $\frac{2}{3}$ and $\frac{4}{5}$.

Since $3 > 0$ and $5 > 0$ and $2 \cdot 5 < 3 \cdot 4$, we conclude $\frac{2}{3} < \frac{4}{5}$.

In the definition above and in all of our examples, we have demanded that the denominators of the fractions used in comparing rational numbers be positive. Is this necessary?

Consider the rational numbers $\frac{2}{1}$ and $\frac{3}{1}$. We have already agreed that $\frac{2}{1} < \frac{3}{1}$, since $1 > 0$ and $2 < 3$. And yet if we were to use the fractions $\frac{-2}{-1}$ and $\frac{-3}{-1}$ to represent these numbers, it is not true that $-2 < -3$. This illustrates the importance of using fractions "$\frac{a}{b}$" and "$\frac{c}{d}$" with b and d positive when comparing rational numbers.

Questions: Can every rational number be represented by a fraction "$\frac{a}{b}$" with $b > 0$? What fraction "$\frac{a}{b}$" where $b > 0$ represents the same rational number as $\frac{3}{-2}$? as $\frac{-7}{-3}$?

12.17 Exercises

1. Represent the rational numbers in each pair below by the fractions having the same denominator. Then decide which rational number is smaller.

   (a) $\frac{2}{5}$ and $\frac{3}{8}$
   (c) $\frac{5}{4}$ and $\frac{7}{5}$
   (b) $\frac{3}{4}$ and $\frac{5}{8}$
   (d) $\frac{8}{3}$ and $\frac{9}{4}$

2. Draw a number line, and locate points on it to represent each of the rational numbers in Exercise 1.

3. Decide which of the following statements are true, and which are false. (As with the integers, the sign ">" means "is greater than.")

   (a) $\frac{7}{5} < \frac{3}{2}$
   (d) $\frac{-2}{7} < \frac{-3}{8}$
   (g) $\frac{-3}{2} < \frac{-7}{4}$

(b) $\frac{1}{2} > \frac{5}{7}$

(c) $\frac{23}{15} < \frac{15}{10}$

(e) $\frac{0}{1} > \frac{5}{7}$

(f) $\frac{-2}{7} < \frac{0}{1}$

(h) $\frac{2}{8} > \frac{1}{4}$

(i) $\frac{85}{32} < \frac{31}{12}$

4. For each pair of rational numbers below, decide which is less.

(a) $\frac{1}{2}, \frac{5}{8}$

(b) $\frac{-1}{2}, \frac{-5}{8}$

(c) $\frac{11}{23}, \frac{7}{15}$

(d) $\frac{-11}{23}, \frac{-7}{15}$

(e) $\frac{8}{-3}, \frac{14}{-5}$

(f) $\frac{12}{9}, \frac{4}{3}$

(g) $\frac{100}{51}, \frac{13}{7}$

(h) $\frac{0}{1}, \frac{0}{52}$

(i) $\frac{5}{-3}, \frac{-7}{4}$

5. If $\frac{a}{b} > \frac{0}{1}$, then $\frac{a}{b}$ is a positive rational number.

If $\frac{a}{b} < \frac{0}{1}$, then $\frac{a}{b}$ is a negative rational number.

Decide whether each of the following rational numbers is positive, negative, or zero.

(a) $\frac{5}{2}$

(b) $\frac{-5}{2}$

(c) $\frac{5}{-2}$

(d) $\frac{-5}{-2}$

(e) $\frac{12}{-7}$

(f) $\frac{-3}{-4}$

(g) $\frac{-11}{5}$

(h) $\frac{4}{3}$

(i) $\frac{0}{8}$

(j) $\frac{10}{-3}$

(k) $\frac{-3}{-10}$

(l) $\frac{1211}{315}$

6. If $\frac{a}{b}$ is a rational number, and the product of the integers $\underline{a}$ and $\underline{b}$ is a positive integer, is the rational number $\frac{a}{b}$ positive? Give an argument for your answer.

7. Answer each of the following, and give an argument for your answer.

(a) Does the ordering of the rational numbers possess the reflexive property?

(b) Does the ordering of the rational numbers possess the symmetric property?

(c) Does the ordering of the rational numbers possess the transitive property?

8. Complete the following sentences for b and d positive integers, and a and c integers.

(a) If $\frac{a}{b} < \frac{c}{d}$, then ad___bc.

(b) If $\frac{a}{b} = \frac{c}{d}$, then ad___bc.

(c) If $\frac{a}{b} > \frac{c}{d}$, then ad___bc.

9. (a) Is there an integer "between" 2 and 3? That is, is there an integer x such that $2 < x$ and $x < 3$? If so, name one.

(b) Is there a rational number between 2 and 3? If so, name one.

(c) Name a rational number between $\frac{2}{3}$ and $\frac{3}{4}$.
(Hint: You might find the "average" of the numbers.)

(d) Name a rational number between $\frac{5}{4}$ and $\frac{7}{5}$.

(e) Given any two rational numbers, do you think it is possible to find another rational number that is between them? Give an argument for your answer.

In problems 10 - 30 make the indicated rational number computations.

10. $\frac{1}{2} \cdot \frac{1}{3}$

11. $3 + \frac{7}{8}$

21. $-2 \cdot (\frac{3}{8} \cdot 8)$

22. $(-2 \cdot \frac{3}{8}) \cdot 8$

12. $7 \cdot \frac{2}{7}$

13. $7 \cdot \frac{7}{2}$

14. $\frac{3}{8} - 2$

15. $\frac{-3}{8} - 2$

16. $\frac{-3}{8} + 2$

17. $2 + \frac{-3}{8}$

18. $(\frac{3}{4} \cdot 9) \cdot \frac{6}{7}$

19. $2 + (\frac{1}{2} + 3)$

20. $(2 + \frac{1}{2}) + 3$

23. $21 - \frac{3}{4}$

24. $\frac{3}{4} - 21$

25. $\frac{-2}{3} \cdot 3$

26. $\frac{2}{3} \cdot -3$

27. $\frac{5}{4} \cdot 3 \cdot \frac{2}{15} \cdot 10$

28. $(\frac{2}{3} \cdot 6) + 3 + \frac{1}{2}$

29. $(2 + \frac{1}{3}) + (\frac{1}{3} + 2)$

30. $(5 \cdot \frac{3}{5})^2 + (\frac{-2}{3} \cdot 3)$

In each of the problems 31 - 36 decide which of the rational numbers in the pair is smaller .

31. $-3, \frac{-7}{3}$

32. $14, \frac{41}{3}$

33. $\frac{21}{5}, 4$

34. $\frac{-21}{5}, -4$

35. $6, \frac{47}{8}$

36. $1, \frac{999}{1000}$

12.18 Decimal Fractions

In the preceding sections, we have developed the system $(Q,+,\cdot)$. Now we look at another way of naming rational numbers, a way that is based on the idea of place value. You are probably already familiar with the idea of place value; for instance, when we write "3607" we mean

$$(3 \cdot 1000) + (6 \cdot 100) + (0 \cdot 10) + (7 \cdot 1), \text{ or}$$
$$(3 \cdot 10^3) + (6 \cdot 10^2) + (0 \cdot 10^1) + (7 \cdot 1).$$

This form is referred to as "expanded notation."

In your elementary school work, you probably saw charts like the one below which explain the place value scheme used in writing names of rational numbers that are also whole numbers.

1	5	4	8	7	6	3
10^6	10^5	10^4	10^3	10^2	10	1
MILLIONS	HUNDRED THOUSANDS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	ONES

Thus, in "1,548,763" the "7" represents 7 hundreds (that is, 700), since it is in the "third place" to the left of the decimal point. (In writing the name of a whole number, it is not common to mark the decimal point, but it is at the extreme right.) There is a very important pattern in this place value scheme. As you move from left to right, the value associated with each place is $\frac{1}{10}$ of the value associated with the preceding place. Thus, with the third place we associate the value 100; but with the second place, we associate the value $\frac{1}{10} \cdot 100$ or 10. In order to have names for all rational numbers (not just whole numbers) we extend this pattern to the right of the decimal point. That is, the value of the first place to the right of the decimal point is $\frac{1}{10} \cdot 1$, or $\frac{1}{10}$; the value of the second place to the right of the decimal point is $\frac{1}{10} \cdot \frac{1}{10}$ or $\frac{1}{100}$. We may also indicate $\frac{1}{100}$ as $\frac{1}{10^2}$. The table below shows the values associated with the first six places to the right of the decimal point. (You should be able to extend the table as far to the right as desired.)

3	4	0	7			
$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$	$\frac{1}{10^6}$	
TENTHS	HUNDREDTHS	THOUSANDTHS	TEN THOUSANDTHS	HUNDRED THOUSANDTHS	MILLIONTHS	

In the table you see the numeral ".3407." The table makes it easy to see that this means

$$(3 \cdot \frac{1}{10}) + (4 \cdot \frac{1}{100}) + (0 \cdot \frac{1}{1000}) + (7 \cdot \frac{1}{10000}).$$

But this is also

$$\frac{3000}{10000} + \frac{400}{10000} + \frac{0}{10000} + \frac{7}{10000} = \frac{3407}{10000}.$$

(Do you see why?)

Therefore,

$$.3407 = \frac{3407}{10000};$$

and ".3407" is a decimal fraction name for a rational number.

Question: Can you write an equation of form "b · x = a" where a and b are integers, whose solution is the rational number .3407?

If you are not already familiar with decimal fraction notation, the following examples should help to make it clear.

Example 1: ".25" is the name of a rational number. Represent this rational number by an irreducible fraction.

We know that $.25 = (2 \cdot \frac{1}{10}) + (5 \cdot \frac{1}{100})$

$$= \frac{20}{100} + \frac{5}{100}$$

$$= \frac{25}{100} .$$

Of course, "$\frac{25}{100}$" is not an irreducible fraction.

But we know that $\frac{25}{100} = \frac{1}{4}$. Therefore,

$$.25 = \frac{1}{4} .$$

Example 2: Represent the rational number .250 by an irreducible fraction.

$$.250 = \frac{250}{1000} = \frac{25}{100} .$$

Do you see then that this example is really the same as Example 1? Again, the irreducible fraction called for is "$\frac{1}{4}$." That is,

$$.250 = .25 = \frac{1}{4} .$$

On the basis of Example 2, you should begin to see why it is true that some rational numbers have an infinite number of decimal fraction representations. Thus,

$$\frac{1}{4} = .25 = .250 = .2500 = .25000, \text{ etc.}$$

Question: Do the decimal fractions ".4" and ".400" represent the same number? Why or why not?

Example 3: Represent the number 4.18 by a fraction $\frac{a}{b}$, where a and b are integers.

$$4.18 = 4 + \frac{18}{100}\ .\quad \text{But } 4 = \frac{400}{100}\ .$$

$$\text{So, } 4.18 = \frac{400}{100} + \frac{18}{100}$$

$$= \frac{418}{100}\ .$$

Example 4. Represent the rational number $\frac{2}{5}$ by a decimal fraction. We know $\frac{2}{5} = \frac{4}{10}$. (Why?) Therefore,

$\frac{2}{5} = .4$. Of course, we could also use ".40," ".400," ".4000," etc.

Example 5: Represent $15\frac{2}{5}$ by a decimal fraction. An expression such as "$15\frac{2}{5}$" is sometimes called a mixed numeral, since it looks as though it is composed of a symbol for a whole number together with a fraction. The important point to understand is that it means

$$15 + \frac{2}{5}\ .$$

Therefore, from Example 4 we know:

$$15\frac{2}{5} = 15 + .4$$

$$= 15.4$$

Example 6 Represent $\frac{3}{8}$ by a decimal fraction. We know that $\frac{3}{8}$ is a quotient; namely, $3 \div 8$. Therefore, in the space at the right we carry out this division. Another way to think about this division is as follows:

$$\begin{array}{r} .375 \\ 8\overline{)3.000} \\ \underline{2.4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \end{array}$$

$$1000 \cdot \frac{3}{8} = \frac{3000}{8}$$

$$= 375.$$

Then, since $1000 \cdot \frac{3}{8} = 375$, $\frac{3}{8} = \frac{375}{1000}$. (Do you remember how a rational number was defined as the solution of an equation?)

12.19 Exercises

1. Express each of the following decimal fractions as an irreducible fraction "$\frac{a}{b}$."

(a) .3
(b) .32
(c) .320
(d) .325
(e) 7.3
(f) .03
(g) .003
(h) .000003
(i) .500
(j) .005
(k) 3.05
(l) 25.1
(m) .625
(n) 10.625
(o) .33

2. We know that every rational number is the solution of an equation of the form "$b \cdot x = a$," where a and b are integers, $b \neq 0$. For each of the following rational numbers, write an equation of which the number is the solution.

Example: $.19 = \frac{19}{100}$

Therefore, .19 is the solution of "$100 \cdot x = 19$."

(a) .5
(b) .7
(c) .08
(d) .07
(e) .33
(f) .333
(g) 2.7
(h) .375
(i) .60
(j) .6
(k) .123456
(l) .333333
(m) -.5
(n) -.05
(o) -2.7
(p) -.375

3. Find a decimal fraction name for each of the following rational numbers. (The rational numbers listed in this

exercise are so frequently used that it is advisable to remember their decimal fraction representations.)

(a) $\frac{1}{2}$
(b) $\frac{1}{4}$
(c) $\frac{3}{4}$
(d) $\frac{1}{5}$
(e) $\frac{2}{5}$
(f) $\frac{3}{5}$
(g) $\frac{4}{5}$
(h) $\frac{1}{8}$
(i) $\frac{3}{8}$
(j) $\frac{5}{8}$
(k) $\frac{7}{8}$

4. For each of the following rational numbers, write four other decimal fractions which represent the same number.

(a) .5
(b) .3
(c) .05
(d) 25.6
(e) 4.0
(f) .025
(g) .000005
(h) .25
(i) 5

5. Recall that a rational number can be represented as a quotient "$\frac{a}{b}$," where a and b, the numerator and denominator, are integers.

(a) In the decimal fraction ".5," what is the numerator? What is the denominator?
(b) What are the numerator and denominator of ".00007"?
(c) What are the numerator and denominator of "8.2"?
(d) Does every decimal fraction represent a rational number? Explain. (How is the numerator determined? How is the denominator determined?)

6. Find a decimal fraction which represents each of the following rational numbers. (See Example 6 of Section 12.18.)

(a) $\frac{3}{20}$
(d) $\frac{21}{25}$

(b) $\frac{7}{20}$ (e) $\frac{25}{64}$

(c) $\frac{8}{25}$ (f) $\frac{63}{200}$

12.20 Infinite Repeating Decimals

Can every rational number be represented by a decimal fraction? The exercises in the preceding section may lead you to answer "yes," and although this is correct, there is a major difficulty with many rational numbers. As an example, let us try to find a decimal fraction for $\frac{1}{3}$. As before, we know this is a quotient, and the appropriate division is shown below:

$$
\begin{array}{r}
\underline{.3333}\ldots \\
3\,|\,1.0000 \\
\underline{9} \\
10 \\
\underline{9} \\
10 \\
\underline{9} \\
10 \\
\underline{9} \\
1
\end{array}
$$

Do you see the difficulty? In this case, the division process is something like a broken record. For as long as we care to continue writing, we will have to place a "3" in each place to the right of the decimal point. Thus this decimal does not "end" or "terminate" as it does, for example, with $\frac{3}{8} = .375$. (See Example 6 of Section 12.18.)

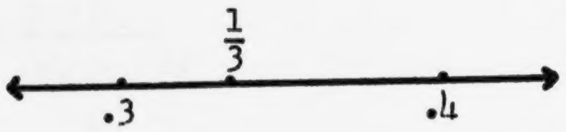
How then can we represent $\frac{1}{3}$ with a decimal fraction? One answer lies in giving an approximate decimal fraction. To see this, study the following steps.

$$0 \leq \frac{1}{3} \leq 1$$

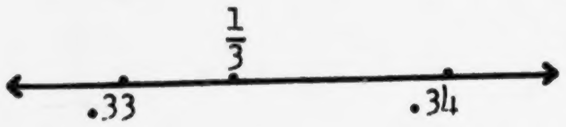
We know that $\frac{1}{3}$ is "between" 0 and 1, and we say that $\frac{1}{3}$ is in the closed interval $\left[0,1\right]$. In terms of a number line this means that the point representing $\frac{1}{3}$ lies on that part of the line consisting of the points representing 0 and 1, together with all the points between those two:

We can also place $\frac{1}{3}$ in smaller and smaller intervals, as follows:

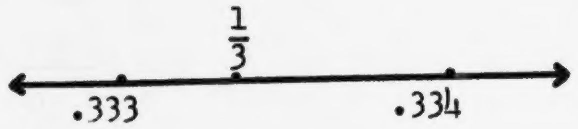
$.3 \leq \frac{1}{3} \leq .4$



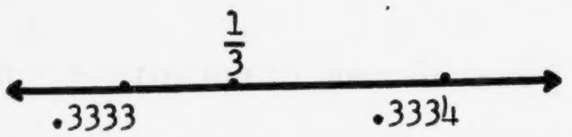
$.33 \leq \frac{1}{3} \leq .34$



$.333 \leq \frac{1}{3} \leq .334$



$.3333 \leq \frac{1}{3} \leq .3334$



Do you see that in a way we are "squeezing" the number $\frac{1}{3}$? Each of the above intervals is smaller than the one before it, and is contained in it. We call such intervals nested intervals. Thus, we have a sequence of nested intervals containing the rational number $\frac{1}{3}$. Although we stopped with the interval $\left[.3333, .3334\right]$, the sequence goes on without end.

Question: Continuing in the pattern above, what is the next interval in this sequence of nested intervals?

If we form a sequence of the left end points of these nested intervals, we get: .3, .33, .333, .3333, .33333, ..., a sequence of rational numbers. None of the numbers in this sequence is equal to $\frac{1}{3}$. For instance, consider the first number, .3:

$.3 \neq \frac{1}{3}$. In fact, $.3 < \frac{1}{3}$. We can find the difference between $\frac{1}{3}$ and .3 as follows:

$$\begin{aligned} \frac{1}{3} - .3 &= \frac{1}{3} - \frac{3}{10} \\ &= \frac{10}{30} - \frac{9}{30} = \frac{1}{30}. \end{aligned}$$

Therefore, although $.3 \neq \frac{1}{3}$, it is "very close" to $\frac{1}{3}$, because the difference between the numbers is small. We can say that .3 is an approximation to $\frac{1}{3}$, and write:

$$\frac{1}{3} \approx .3$$

This approximation is said to be correct to tenths or to one decimal place.

Next let us consider the second number in the sequence, .33. The difference between this number and $\frac{1}{3}$ is computed below:

$$\begin{aligned} \frac{1}{3} - .33 &= \frac{1}{3} - \frac{33}{100} \\ &= \frac{100}{300} - \frac{99}{300} \\ &= \frac{1}{300}. \end{aligned}$$

Therefore .33 is a better approximation to $\frac{1}{3}$ than is .3. That is, it is closer to $\frac{1}{3}$ since it differs from it by only $\frac{1}{300}$ instead of $\frac{1}{30}$. (How do we know that $\frac{1}{300} < \frac{1}{30}$?) Thus we write

$$\frac{1}{3} \approx .33,$$

and say that this approximation is correct to hundredths or to two decimal places.

In fact, as you might have guessed, each number in the sequence

$$.3, \quad .33, \quad .333, \quad .3333, \ldots$$

is a closer approximation to $\frac{1}{3}$ than the number preceding it.

Question: What is the difference between $\frac{1}{3}$ and .333? Though we shall not explore the matter here, it is true that by going far enough in the sequence you can get a number as close to $\frac{1}{3}$ as you like.

Now, from the number $\frac{1}{3}$, we have learned a very important fact. Not every rational number can be expressed by a terminating decimal fraction. Many rational numbers, such as $\frac{1}{3}$, have decimal fraction representations that are infinite, repeating decimals.

As another example, let us work with the rational number

$$\frac{8}{33}.$$

$$\begin{array}{r}
.2424\ldots \\
33\,\overline{)\,8.0000} \\
\underline{6\ 6} \\
1\ 40 \\
\underline{1\ 32} \\
80 \\
\underline{66} \\
140 \\
\underline{132} \\
8
\end{array}$$

$$.24 < \frac{8}{33} < .25$$

$$.2424 < \frac{8}{33} < .2425$$

$$.242424 < \frac{8}{33} < .242425$$

$\frac{8}{33} \approx .24$ (correct to hundredths)

$\frac{8}{33} \approx .2424$ (to four decimal places)

$\frac{8}{33} \approx .\,242424$

As with $\frac{1}{3}$, there is no terminating decimal representation for $\frac{8}{33}$, but there is an infinite repeating decimal associated with it, and we can approximate $\frac{8}{33}$ to any desired number of decimal places.

12.21 Exercises

1. (a) What is the difference between $\frac{1}{3}$ and .333?

   (b) What is the difference between $\frac{1}{3}$ and .3333?

   (c) Which of the numbers, .333 and .3333, is a better approximation to $\frac{1}{3}$?

2. (a) Write an equation of the form " $b \cdot x = a$," where a and b are integers, which has $\frac{1}{3}$ as solution.

   (b) Write an equation of the form " $b \cdot x = a$," where a and b are integers, which has .3 as solution.

(c) Write an equation of the form "$b \cdot x = a$ where a and b are integers, which has .33 as solution.

(d) Would the same equation work for all of the parts (a), (b), and (c)? Why or why not?

3. In looking for a decimal fraction representation of $\frac{1}{6}$, the division process below might be used:

$$\begin{array}{r}
\overline{.1666}\ldots \\
\underline{6}\,|\,1.0000 \\
\underline{6} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
4
\end{array}$$

Thus, we again get an infinite repeating decimal, although the digits do not start repeating right away.

(a) What is the difference between $\frac{1}{6}$ and .16?

(b) What is the difference between $\frac{1}{6}$ and .17?

(c) Which is a better approximation to $\frac{1}{6}$, .16 or .17?

(d) What is the difference between $\frac{1}{6}$ and .166?

(e) What is the difference between $\frac{1}{6}$ and .167?

(f) Which is a better approximation to $\frac{1}{6}$, .166 or .167?

(g) Which is a better approximation to $\frac{1}{6}$, .17 or .167?

(h) What is the best approximation to $\frac{1}{6}$, correct to four decimal places?

4. For each of the following rational numbers, write the best decimal fraction approximation, correct to four decimal places.

(a) $\frac{5}{6}$ (c) $\frac{1}{11}$ (e) $\frac{1}{12}$

(b) $\frac{2}{3}$ (d) $\frac{2}{11}$ (f) $\frac{5}{12}$

5. Consider the sequence below:

.1, .11, .111, .1111, ...

(a) What is the difference between $\frac{1}{9}$ and .1?

(b) What is the difference between $\frac{1}{9}$ and .11?

(c) What is the difference between $\frac{1}{9}$ and .111?

(d) What is the difference between $\frac{1}{9}$ and .1111?

(e) Suppose the sequence continues in the pattern suggested by the first four terms. How far would you have to go in the sequence to find a number that differs from $\frac{1}{9}$ by $\frac{1}{9{,}000{,}000}$?

6. (a) Give an approximate decimal fraction (correct to three decimal places) for the rational number

$$2\frac{1}{3} = \frac{7}{3}.$$

(b) Is the decimal fraction representation of $2\frac{1}{3}$ an infinite repeating decimal? (Remember that the decimal fraction need not start repeating right away.)

7. Consider the number $\frac{1}{7}$.

   (a) In dividing by 7, how many numbers are possible as remainders? (Remember that a remainder must be less than the divisor.)

   (b) Carry out the division process for $1 \div 7$ to twelve decimal places.

   (c) At what stage in the division process did you get a remainder that had occurred before?

   (d) At what stage in the division process did the decimal fraction start "repeating"? Can you explain why it happened at that particular time?

8. In carrying out the division $3 \div 8$, what remainder occurs that causes the decimal fraction to terminate?

9. Try to give a convincing argument for the following:

   The decimal fraction representation for any rational number $\frac{a}{b}$ is either a terminating decimal or an infinite repeating decimal.

10. Write a sequence of nested intervals all of which contain the number $\frac{1}{11}$. Begin with the interval $[0,1]$ and get a total of five intervals. Also show the intervals on a number line.

11. Explain why the following sequence of intervals is not a nested sequence:

$$[0,1], \quad [1,2], \quad [1.5,2.5], \quad [.1,.2]$$

12.22 Decimal Fractions and Order of the Rational Numbers

We have already seen how to tell which of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is less, when fractions are used to represent the numbers. Now let us see how to make such a comparison when decimal fractions are used.

Example 1: Which is less, .3 or .4?

Since $.3 = \frac{3}{10}$ and $.4 = \frac{4}{10}$

it is easy to tell that $.3 < .4$.

Example 2: Which is less, .2567 or .2563?

Notice that first three digits of these decimal fractions agree, place by place. The fourth decimal place is the first one in which they differ.

$$.2567 = \frac{256}{1000} + \frac{7}{10000};$$

$$.2563 = \frac{256}{1000} + \frac{3}{10000}.$$

Therefore, $.2563 < .2567$.

Example 3: Which is less, .8299 or .8521?

$$.8299 = \frac{8}{10} + \frac{299}{10000}$$

$$.8521 = \frac{8}{10} + \frac{521}{10000}$$

Therefore, $.8299 < .8521$.

Notice again that these two decimal fractions agree in the first decimal place. The first place in which they disagree is the second place; and $2 < 5$.

These three examples show that it is very easy to tell which of the two rational numbers is less when the numbers are represented by decimal fractions. Suppose we have two decimal fractions

$$.a_1 a_2 a_3 a_4$$

and

$$.b_1 b_2 b_3 b_4$$

and $a_1 = b_1$, $a_2 = b_2$, but $b_3 < a_3$. Then do you see that $.b_1 b_2 b_3 b_4 < .a_1 a_2 a_3 a_4$? In other words, the way to tell which of two decimal fractions represents the smaller number is to look for the first place (reading from left to right) in which they disagree; the one which has the smaller entry in that place represents the smaller number.

Example 4: Which is less, 23.524683 or 23.524597? The first place in which these decimal fractions "disagree" is the fourth decimal place. And since $5 < 6$, then $23.524597 < 23.524683$.

12.23 Exercises

1. In each of the following, copy the two rational numbers. Then place either "<" or ">" or "=" between them so that a true statement results.

(a) 12.5 12.4
(b) 8.33 8.34
(c) .1257 .1250
(d) .1257 .125
(e) .6666 .6667
(f) 826.33 826.30
(g) 5.4793293 5.4789999
(h) 548 551
(i) 1.9999 2
(j) .9874 .9875

2. This exercise is similar to Exercise 1, except that

negative rational numbers are used. Remember that although $1 < 2$, for instance, $-2 < -1$. Thus, although $.5 < .6$, we have $-.6 < -.5$.

(a) -3.567   -3.582

(b) -.12345   -.12453

(c) -.99   -1

(d) -100.555   -100.565

(e) -42.80   -42.85

(f) -42.8   -42.85

(g) -12.9999   -12.9998

(h) -4.378   -4.3779

3. Is it possible to find a rational number x "between" .354 and .357? That is, we want a number x such that

$$.354 < x < .357.$$

Notice that these two decimal fractions agree in the first two places, but disagree in the third place. Thus, for x, we can use a decimal fraction that agrees with the two given ones in the first two places, but has in the third place a digit that is between the two given third digits. For example, x might be .355, since $.354 < .355 < .357$. (This is not the only value of x that can be used. Can you give others?)

Now for each pair of rational numbers below, name a rational number that is between them.

(a) .6, .8

(b) 2.35, 2.39

(c) 45.987, 45.936

(d) 102, 108

(e) 5.420, 5.430

(f) 5.42, 5.43

(g) 3.8, 3.9

(h) 2.99, 3

Compare Exercise 3 here with Exercise 9 in Section 12.17. Do you see that between two rational numbers it is always possible to find another rational number? For this reason, we say that $(Q,<)$ is dense: that is, the rational numbers form a *dense set*.

4. Given the rational numbers 1 and 2 find a rational number x such that $1 < x < 2$; then find a rational number y such that $1 < y < x$; then find a rational number z such that $1 < z < y$; then find a rational number w such that $1 < w < z$.
   Draw a number line and represent the numbers $1, 2, x, y, z, w$, by points on the scale.

5. Do the integers form a dense set? Why or why not?

12.24 Summary

In this chapter we have developed the *rational number system*. In order to see why this system is such an important one, let us retrace some of the steps in its development.

In the *whole number system*, there are two binary operations, addition and multiplication. Subtraction and division are *not* operations. Thus, for example, the subtraction $2 - 5$ and the division $2 \div 5$ are not possible in $(W,+,\cdot)$. We might say that subtraction and division are "deficiencies" of the whole number system. Part of our work this year has been concerned with removing these deficiencies.

We first removed the subtraction deficiency by developing $(Z,+,\cdot)$, the number system of *integers*. Subtraction is a binary operation in this system, $2 - 5$, for example, is -3. And since $(Z,+,\cdot)$ contains $(W,+,\cdot)$, we have in the integers all of the operations and properties of W, together with the new operation of subtraction. Thus, Z is an "extension" of W, a fact suggested by the following diagram:

$$\begin{tikzcd}
\boxed{(W, +, \cdot)} \arrow[r, dashed, "\text{extension to}", "\text{make subtraction an operation}"'] & \boxed{(Z, +, \cdot)}
\end{tikzcd}$$

However, division is not an operation on Z, and in this chapter we removed this deficiency by developing the system (Q,+,·) in which division (except by 0) is always possible. For example, the quotient 2 ÷ 5 is the rational number we have called $\frac{2}{5}$. And since (Q,+,·) contains (Z,+,·), Q is an extension of Z. Therefore, we can complete the above diagram as follows:

$$\begin{tikzcd}
\boxed{(W, +, \cdot)} \arrow[r, dashed, "\text{extension to}", "\text{make subtraction an operation}"'] & \boxed{(Z, +, \cdot)} \arrow[d, dashed, "\text{extension division}"', "\text{to make an operation}"] \\
 & \boxed{(Q, +, \cdot)}
\end{tikzcd}$$

In (Q,+,·) the four operations are defined as follows, $(b,d \neq 0)$:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \quad (c \neq 0)$$

(Q,+,·) has the following important properties.

If x, y, and z are rational numbers, then:

$$(x + y) + z = x + (y + z) \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$x + 0 = x \qquad x \cdot 1 = x$$

$$x + (-x) = 0 \qquad x \cdot \frac{1}{x} = 1 \ (x \neq 0)$$

$$x + y = y + x \qquad x \cdot y = y \cdot x$$

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Every equation "$px = q$" where p and q are rational numbers, and $p \neq 0$, has a solution, namely $x = q \div p$, in $(Q,+,\cdot)$

The rational numbers are ordered. If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, with b and d both positive, then

$$\frac{a}{b} < \frac{c}{d} \text{ if and only if } ad < bc.$$

The rational numbers form a dense set. Between any two different rational numbers, there is another rational number.

12.25 Review Exercises

1. Solve the following equations.

(a) $4 \cdot x = 3$
(b) $3 \cdot x = 4$
(c) $-4 \cdot x = 3$
(d) $4 \cdot x = -3$
(e) $-4 \cdot x = -3$
(f) $12 \cdot x = 5$
(g) $3 \cdot x = 20$
(h) $3 \cdot x = 21$
(i) $7 \cdot x = 5$
(j) $-3 \cdot x = 8$
(k) $102 \cdot x = 511$
(l) $-55 \cdot x = 30$
(m) $87 \cdot x = 87$
(n) $87 \cdot x = 0$
(o) $4 \cdot x = a$

2. Compute the following.

(a) $\frac{2}{3} + \frac{3}{4}$

(b) $\frac{2}{3} + \frac{3}{4}$

(c) $\frac{5}{2} - \frac{4}{7}$

(i) $\frac{9}{4} + \frac{5}{6}$

(j) $\frac{2}{3} \cdot \frac{5}{5}$

(k) $8 - \frac{5}{4}$

(d) $\frac{4}{7} - \frac{5}{2}$

(e) $\frac{8}{5} \cdot \frac{8}{5}$

(f) $\frac{8}{5} \cdot \frac{5}{8}$

(g) $\frac{1}{2} \div \frac{3}{8}$

(h) $\frac{3}{8} \div \frac{1}{2}$

(l) $8 \div \frac{5}{8}$

(m) $\frac{5}{4} - 8$

(n) $\frac{5}{4} \div 8$

(o) $3 \div 7$

(p) $7 \div 3$

3. Compute the following.

(a) $(\frac{1}{2} + \frac{3}{4}) + \frac{7}{8}$

(b) $\frac{2}{3}(\frac{1}{2} + \frac{3}{5})$

(c) $\frac{0}{4} + (\frac{9}{5} + \frac{3}{10})$

(d) $\frac{10}{3} \div (\frac{3}{4} \div \frac{1}{2})$

(e) $(\frac{10}{3} \div \frac{3}{4}) \div \frac{1}{2}$

(f) $(\frac{3}{8} + \frac{5}{6}) \div \frac{2}{5}$

(g) $\frac{3}{8} + (\frac{5}{6} \div \frac{2}{5})$

(h) $(8 \div \frac{1}{3}) - \frac{1}{10}$

(i) $\frac{4}{3} \cdot \frac{3}{16} \cdot \frac{3}{4} \cdot \frac{5}{9} \cdot \frac{9}{5} \cdot \frac{16}{3}$

(j) $\frac{-2}{3} + \frac{7}{5} + \frac{-7}{5} + \frac{0}{1} + \frac{2}{3} + \frac{1}{2}$

4. Compute the following:

(a) $\frac{3}{4} \div \frac{7}{8}$

(b) $\frac{9}{2} \div \frac{2}{9}$

(c) $\frac{14}{3} \div \frac{7}{5}$

(d) $\frac{12}{5} \div \frac{3}{8}$

(e) $\frac{a}{b} \div \frac{c}{d}$

5. Write each of the following in expanded notation.

Example: $.23 = (2 \cdot \frac{1}{10}) + (3 \cdot \frac{1}{100})$

(a) .6

(b) .63

(c) .063

(d) .00603

(e) 25.08

(f) 3.175

(g) 2.000005

(h) .3333

6. Write a decimal fraction representation of each of the following. If the decimal does not terminate, give an approximation to four decimal places (i.e., correct to ten thousandths).

(a) $\frac{1}{2}$

(b) $\frac{13}{26}$

(c) $\frac{3}{4}$

(d) $\frac{2}{5}$

(e) $3\frac{2}{5}$

(f) $\frac{1}{3}$

(g) $\frac{7}{10}$

(h) $\frac{70}{100}$

(i) $\frac{5}{8}$

(j) $\frac{1}{7}$

7. Copy the following, and place one of the three symbols, "<," ">" or "=" between the pairs of rational numbers so that a true statement results in each case.

(a) $\frac{1}{2}$ $\frac{2}{3}$

(b) $\frac{4}{7}$ $\frac{5}{9}$

(c) $\frac{23}{5}$ $\frac{25}{7}$

(d) .3475 .3429

(e) $\frac{1}{3}$ .333333

(f) .375 $\frac{3}{8}$

(g) .00001 .000009

(h) $\frac{20}{7}$ $\frac{25}{12}$

(i) $\frac{-3}{5}$ $\frac{-2}{3}$

8. For each pair of rational numbers below, write the name of a rational number that is between them.

(a) $\frac{1}{2}$, 1

(b) $\frac{1}{2}$, $\frac{3}{4}$

(c) $\frac{1}{2}$, $\frac{5}{8}$

(d) $\frac{1}{2}$, $\frac{17}{32}$

(e) $\frac{1}{3}, \frac{4}{9}$

(f) .345, .346

(g) $\frac{7}{3}, \frac{13}{5}$

(h) 0, $\frac{1}{100}$

(i) 0, .000001

9. Solve the following equations.

(a) $\frac{2}{3} \cdot x = \frac{3}{5}$

(b) $\frac{2}{3} + x = \frac{3}{5}$

(c) $x \cdot \frac{4}{3} = \frac{1}{2}$

(d) $\frac{7}{2} + x = \frac{-4}{5}$

CHAPTER 13

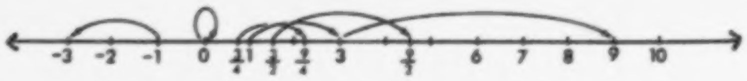
SOME APPLICATIONS OF THE RATIONAL NUMBERS

13.1 Rational Numbers and Dilations

Earlier, you learned that "D_{ab}" means "$D_b \circ D_a$," or the dilation D_a followed by the dilation D_b. At that time, it was required that a and b be integers. Let us now consider the composition $D_b \circ D_a$ where a and b are rational numbers. For the present we shall restrict the discussion to dilations on a line. Consider

$$D_{\frac{1}{2}} \circ D_3.$$

Since D_3 acts first, we show below the images of certain points under this dilation.

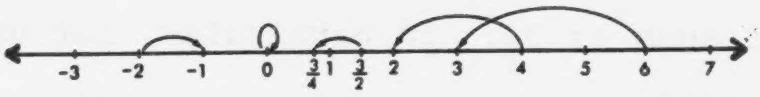


Since we are using the rational numbers, any point with a rational coordinate has an image under this dilation. For instance, the point with coordinate $\frac{3}{4}$ is mapped into the point with coordinate $\frac{9}{4}$, since $3 \cdot \frac{3}{4} = \frac{9}{4}$.

Question: Under dilation D_3, what are the coordinates of the images of the points having the following coordinates:

$\frac{1}{3}$, 1, $\frac{2}{3}$, 10, 100, -1, $-\frac{1}{3}$.

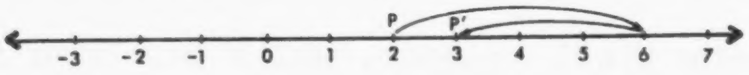
In order to be consistent with the way in which we interpreted D_a, where a is an integer, we shall say that under $D_{\frac{1}{2}}$ a point P with coordinate x is mapped into a point P' whose coordinate is $\frac{1}{2} \cdot x$. The images of certain points under the dilation $D_{\frac{1}{2}}$ are shown below.



Question: Under the dilation $D_{\frac{1}{2}}$, what are the coordinates of the images of the points having the following coordinates:

$1, 2, \frac{1}{2}, \frac{3}{2}, 10, 100, -2.$

Now we consider the compositions $D_{\frac{1}{2}} \circ D_3$. The diagram below shows the image (under this composition) of the point with coordinate 2.



Note that D_3 takes 2 into 6, then $D_{\frac{1}{2}}$ takes 6 into 3. Thus $(D_{\frac{1}{2}} \circ D_3)$ takes 2 into 3. Generally, under the composition $D_{\frac{1}{2}} \circ D_3$, any point P has an image P' whose distance from the origin is $\frac{3}{2}$ times the distance of the point P from the origin. In other words, we may write:

$$D_{\frac{1}{2}} \circ D_3 = D_{\frac{3}{2}}$$

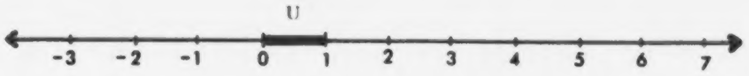
Thus we see that the dilation $D_{\frac{3}{2}}$ may be considered as the composition of two dilations, the first D_3, the second $D_{\frac{1}{2}}$.

Question: What is $D_3 \circ D_{\frac{1}{2}}$? Explain why $D_a \circ D_{\frac{1}{b}} = D_{\frac{1}{b}} \circ D_a$.

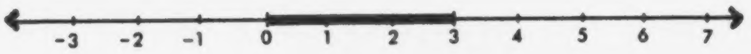
Question: Since under $D_{\frac{3}{2}}$ the image of any point is $\frac{3}{2}$ as far from the origin as the point itself, what do you think the inverse of $D_{\frac{3}{2}}$ is? Consider $D_{\frac{a}{b}} \circ D_{\frac{3}{2}} = D_1$.

Question: Express $D_{\frac{3}{4}}$ as the composition of two dilations.

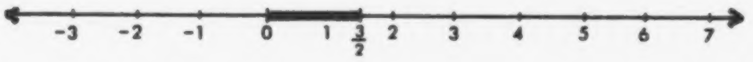
It is also instructive to look at what happens to a segment under a dilation such as $D_{\frac{3}{2}}$. In particular, let us look at the segment whose endpoints are those having coordinates 0 and 1; such a segment is called a unit segment, and we shall denote it by "U."



Now since $D_{\frac{3}{2}}$ is the composition $D_{\frac{1}{2}} \circ D_3$, do you see that segment U is first "stretched" to a segment 3 times as long.



Then, that segment is "shrunk" to a segment half as long, as the diagrams show. The final segment, which has been labeled



V, is then the image of U under the dilation $D_{\frac{3}{2}}$. We may simply write:

$$V = \tfrac{3}{2}U,$$

which may be read "V is $\frac{3}{2}$ times U," or "V is $\frac{3}{2}$ the length of segment U."

Example 1. If a segment X has a length of 10 inches, what is the length of $\frac{3}{4}X$?

We could think of this problem in terms of the composition of dilations $D_{\frac{1}{4}} \circ D_3$ on a line. If the segment X is first "stretched" to a segment 3 times as long, the resulting segment has a length of 30 inches. If that segment is then "shrunk" to a segment one-fourth as long, the length of the resulting segment is $\frac{1}{4} \cdot 30$, or $\frac{30}{4}$ inches. In practice, of course, it is not necessary to explain the solution in this way. We may simply write

$$\frac{3}{4} \text{ of } 10 = \frac{3}{4} \cdot 10 = \frac{30}{4} \text{ (or } \frac{15}{2}\text{)}.$$

Example 2. If segment X has length 10 inches, what is the length of $\frac{4}{3}X$?

$$\frac{4}{3} \text{ of } 10 = \frac{4}{3} \cdot 10 = \frac{40}{3} \text{ inches.}$$

Whole number dilations, D_1, D_2, D_3---are sometimes called stretchers where D_1 is the identity stretcher. The unit fraction dilations $D_{\frac{1}{1}}$, $D_{\frac{1}{2}}$, $D_{\frac{1}{3}}$---are sometimes referred to as shrinkers. $D_{\frac{1}{1}}$ is the same as D_1 or the identity shrinker.

Notice that in Example 1 the final segment is shorter than the segment X, while in Example 2 the final segment is longer than X. Is there any way to predict this beforehand from the dilations $D_{\frac{3}{4}}$ and $D_{\frac{4}{3}}$? (Compare the "stretcher" and "shrinker" in each case.)

Question: How must a and b be related so that under the dilation $D_{\frac{a}{b}}$:

(1) the image of a segment is longer than the segment itself?

(2) the image of a segment is shorter than the segment itself?

(3) the image of a segment is the segment itself.

Answers: (1) $a>b$; (2) $a<b$; (3) $a = b$.

13.2 Exercises

1. Draw three separate number lines, and on each mark points with the following coordinates:

   $0, 1, 2, \frac{5}{2}, \frac{3}{4}$, and -1.

   (a) On one of the drawings, show the image of each of the points marked under the dilation D_2.

   (b) On another of the drawings, show the image of each of images from part (a) under the dilation $D_{\frac{1}{3}}$.

   (c) On the third drawing, show the images of each of the original points under the composition $D_{\frac{1}{3}} \circ D_2$.

   (d) Express the composition of dilations in part (c) as a single dilation.

   (e) Express each of the following as single dilations D_x, where x is a rational number:

   $$D_{\frac{1}{5}} \circ D_4,\ D_{\frac{1}{3}} \circ D_7,\ D_{\frac{1}{2}} \circ D_{10},\ D_{10} \circ D_{\frac{1}{2}}.$$

2. Draw two number lines, and on each mark points with the following coordinates:

   $0, 1, 2, 3, 4, \frac{1}{2}, \frac{8}{3}$, and -2.

   (a) On one drawing, show the image of each of these points under the dilation $D_{\frac{1}{2}}$.

   (b) On another drawing, show the image of each of the original points under the dilation $D_{\frac{2}{4}}$.

(c) Is it correct to write $D_{\frac{1}{2}} = D_{\frac{2}{4}}$?

(d) When is $D_{\frac{a}{b}} = D_{\frac{c}{d}}$?

3. On a number line, let P be the point with coordinate 2.

(a) Let P' be the image of P under $D_{\frac{5}{3}}$. What is the coordinate of P' ?

(b) Let P" be the image of P' under $D_{\frac{2}{3}}$. What is the coordinate of P"?

(c) What is the image of the original point P under the composition $D_{\frac{2}{3}} \circ D_{\frac{5}{3}}$?

(d) Write the composition in part (c) as a single dilation.

4. (a) Write a single dilation $D_{\frac{x}{y}}$ for the composition $D_{\frac{7}{3}} \circ D_{\frac{5}{2}}$.

(b) According to the definition in Chapter 12, what is the product $\frac{7}{3} \cdot \frac{5}{2}$?

In this section, we have used dilations to give meaning to a statement such as $\frac{3}{2}$ of X, where X is a segment. And this kind of expression is common in everyday uses of mathematics. For example, if X represents a class of students, then $\frac{2}{3}$ of X (that is, "$\frac{2}{3}$ of the class") can be interpreted in much the same way as with segments. We really mean $\frac{2}{3}$ times the measure of X. And in this case, the measure is a whole number. Thus, if there are 30 people in the class, "$\frac{2}{3}$ of the class" is 20, since $\frac{2}{3} \cdot 30 = 20$. Problems 5 through 12 are of this kind.

5. There are 100 senators in the United States Senate. On a

recent vote, $\frac{13}{20}$ of the Senate votes "yes" on a certain bill. How many Senators voted "yes"?

6. A certain state has an area of 70,000 square miles. $\frac{3}{100}$ of the state is irrigated land. How many square miles in the state are irrigated?

7. Jim has $2,000 in the bank, and the bank is supposed to pay him $\frac{3}{100}$ of that amount for interest. How much should Jim receive?

8. In 1960, the population of a certain town was 18,000. Today the population is $\frac{5}{3}$ of that number. What is the population today?

9. A family spends $\frac{23}{100}$ of its income on food. If the income for one year is $8500, how much money does this family spend for food in one year?

10. If one pound of ground meat costs $.90 what will be the cost of $2\frac{1}{2}$ pounds?

11. (a) If Jim's height is $\frac{4}{3}$ of Bill's height, who is taller?
    (b) If Mary's height is $\frac{3}{4}$ of Sue's height, who is taller?
    (c) If Bob's height is $\frac{4}{4}$ of John's height, who is taller?

12. In a certain town, there are 5000 registered voters. In a recent election, 3500 people voted. What "fraction" of the town's registered voters actually voted? (Express your answer by an irreducible fraction $\frac{a}{b}$. Check your result by showing that $\frac{a}{b}$ of 5000 is 3500.)

13. In this problem we consider dilations D_x in the plane, where x is a rational number. Just as Z x Z is the set of all points with coordinates (a,b), where a and b are integers,

so Q x Q is the set of all points with coordinates (x,y), where x and y are rational numbers.

(a) Draw a pair of axes, and plot all points whose coordinates are (a,b), where a and b are integers between -4 and 4.

(b) Now plot a point with coordinates $(\frac{3}{2},\frac{7}{4})$. Note that this point does not belong to Z x Z, but it does belong to Q x Q.

(c) Consider the dilation D_2. Under this dilation, the image of $(\frac{3}{2},\frac{7}{4})$ is defined to be $(2\cdot\frac{3}{2},2\cdot\frac{7}{4})$, or $(3,\frac{7}{2})$. Plot this image point. (Do you see a segment in the plane that has been "stretched" to twice its original length?)

(d) Under the dilation $D_{\frac{1}{2}}$, the image of $(\frac{3}{2},\frac{7}{4})$ is $(\frac{1}{2}\cdot\frac{3}{2}, \frac{1}{2}\cdot\frac{7}{4})$. Plot this image point. (Do you see a segment in the plane that has been "shrunk" to $\frac{1}{2}$ of its original length?)

14. From Exercise 13, we make the following definition: If (x,y) is an element of Q x Q, and D_c is a dilation where c is a rational number, then the image of (x,y) under D_c is (cx,cy).

(a) Plot the images of the following points under $D_{\frac{3}{4}}$: (2,8), (4,12), (9,-4), (-8,6), (-2,-12), (0,0), (1,1).

(b) Now for each image from part (a), plot the image of that image under $D_{\frac{4}{3}}$.

(c) How are the dilations $D_{\frac{3}{4}}$ and $D_{\frac{4}{3}}$ related?

15. Show that $D_{\frac{a}{b}} \circ D_{\frac{b}{a}} = D_1$

(a) when $b < a$;

(b) when $b > a$;

(c) when $b = a$.

16. What is the inverse dilation of :

(a) $D_{\frac{2}{3}}$

(b) $D_{\frac{1}{5}}$

(c) $D_{\frac{4}{3}}$

(d) $D_{\frac{x}{y}}$ $(y \neq 0)$

17. (a) How would you describe the images of the points in Q x Q under the dilation D_0?

(b) How would you describe the images of the points in Q x Q under the dilation D_1?

(c) How would you describe the images of the points in Q x Q under the dilation D_{-1}?

13.3 Computation with Decimal Fractions

In Section 13.1 we dealt with such problems as finding "$\frac{3}{4}$ of X." For example if X is a segment having length $2\frac{1}{2}$ inches, then

$$\frac{3}{4} \text{ of } X = \frac{3}{4} \cdot 2\frac{1}{2} = \frac{3}{4} \cdot \frac{5}{2} = 1\frac{7}{8} .$$

At times, problems such as this are expressed in terms of decimal fractions. For instance, we could just as easily speak of finding .75 of a segment X whose length is 2.5 inches. Then we would have to compute

$$.75 \times 2.5.$$

The result should be the same as before, $1\frac{7}{8}$. How is the computation with decimal fractions carried out? Study the computation below.

$$.75 \times 2.5 = \frac{75}{100} \times \frac{25}{10} = \frac{1875}{1000} = 1.875$$

Thus, .75 x 2.5 = 1.875.

This computation could be done as below:

$$\begin{array}{r} 2.5 \\ \underline{\times\ .75} \\ 125 \\ \underline{175} \\ 1.875 \end{array}$$

There is a relationship between the number of digits to the right of the decimal point in the product 1.875, and the number of digits to the right of the decimal point in the two factors, 2.5 and .75. Do you see what the relationship is? (It is a result of the fact that $\frac{1}{100} \times \frac{1}{10} = \frac{1}{1000}$.)

Question: To which of the following is the product

1.5 x 1.5 equal?

.225, 2.25, 22.5, 225.

What is the sum of \$2.45 and \$3.87? The computation is shown below.

$$\begin{array}{r} \$2.45 \\ \underline{+\ \ \$3.87} \\ \$6.32 \end{array}$$

Notice that we add tenth to tenths, hundredths to hundredths, etc.

$$2.45 = 2 + \frac{4}{10} + \frac{5}{100} \text{ ; and}$$

$$3.87 = 3 + \frac{8}{10} \quad \frac{7}{100} .$$

Then,

$$\begin{aligned} 2.45 + 3.87 &= (2 + \frac{4}{10} + \frac{5}{100}) + (3 + \frac{8}{10} \quad \frac{7}{100}) \\ &= (2 + 3) + (\frac{4}{10} + \frac{8}{10}) + (\frac{5}{100} + \frac{7}{100}) \\ &= 5 + \frac{12}{10} + \frac{12}{100} \\ &= 5 + \frac{13}{10} + \frac{2}{100} \qquad (\text{since } \frac{10}{100} = \frac{1}{10}) \\ &= 6 + \frac{3}{10} \quad \frac{2}{100} \qquad (\text{since } \frac{10}{10} = 1) \\ &= 6.32 \end{aligned}$$

In these steps, you should be able to point out where we have used the associative and commutative properties of addition of rational numbers.

Subtraction computations with decimal fractions are done in a way similar to addition computations, as the following example illustrates.

Example 1. Subtract 4.387 from 12.125.

$$\begin{array}{r} 12.125 \\ - \underline{4.387} \\ 7.738 \end{array}$$

(We can check this result by noting that
7.738 + 4.387 = 12.125.)

The quotient of two rational numbers may also be computed when decimal fractions are used to represent the numbers. First, consider the quotient .125 ÷ .5. We may express this quotient as

$$\frac{.125}{.5}$$

and we know this is the same as

$$\frac{.125}{.5} \times \frac{10}{10}. \quad \text{(Why?)}$$

Furthermore, $\frac{.125}{.5} \times \frac{10}{10} = \frac{1.25}{5}$

Therefore, instead of working with the quotient $\frac{.125}{.5}$,

we may compute the equivalent quotient $\frac{1.25}{5}$. The computation is shown below:

$$\begin{array}{r} .25 \\ 5\,\overline{)\,1.25} \\ \underline{1\;0} \\ 25 \\ \underline{25} \end{array}$$

This process is justified by the following:

$$\frac{1.25}{5} = \frac{1}{5} \times 1.25 = \frac{1}{5} \times \left(\frac{1}{100} \times 125\right) = \frac{1}{100} \times \left(\frac{1}{5} \times 125\right)$$

$$= \frac{1}{100} \times 25 = .25.$$

In the preceding division problem we multiplied the given quotient $\frac{.125}{.5}$ by $\frac{10}{10}$ so that we obtained the equivalent quotient $\frac{1.25}{5}$, in which the denominator (divisor) is a whole number. If we try the same approach with the quotient

$$\frac{.0221}{.13},$$

we choose to multiply by $\frac{100}{100}$. (Do you see why?) Thus,

$$\frac{.0221}{.13} = \frac{.0221}{.13} \times \frac{100}{100}$$

$$= \frac{2.21}{13}$$

$$\begin{array}{r} .17 \\ 13\overline{)2.21} \\ \underline{1\ 3} \\ 91 \\ \underline{91} \end{array}$$

$$= .17$$

Therefore, $\frac{.0221}{13} = .17$.

Question: What is the product .17 x .13?

Often, quotients of rational numbers (expressed by decimal fractions), need be carried out only to a specified number of decimal places. Study the example below, in which the quotient has been computed correct to two decimal places (hundredths).

Example 2. What is the quotient when 253.42 is divided by 8.7?

$$\frac{253.42}{8.7} = \frac{253.42}{8.7} \times \frac{10}{10} = \frac{2534.2}{87}$$

$$\begin{array}{r} 29.128 \\ 87\overline{)2534.200} \\ \underline{174} \\ 794 \\ \underline{783} \\ 112 \\ \underline{87} \\ 250 \\ \underline{174} \\ 760 \\ \underline{696} \\ 64 \end{array}$$

Therefore, correct to two decimal places, the quotient is 29.13. That is,

$$\frac{253.42}{8.7} \approx 29.13$$

(The symbol "$\approx$" means "is approximately equal to.") We wrote the answer (correct to 2 decimal places) as 29.13, rather than 29.12, because 29.128 is closer to 29.13 than to 29.12.

Questions: What is the product 29.13 x 8.7? Why is this product not equal to 253.42?

13.4 Exercises

1. Compute the following:

   (a) 2.56 + 8.94

   (b) 10.487 + 35.733

   (c) 42.56 - 387.29

   (d) 4.5 x 2.5

   (e) 2.25 x 2.25

   (f) -3.5 x .4

   (g) -4.85 + -6.15

   (h) 21.5 - (-7.6)

   (i) 55.0 - 39.8

   (j) 39.8 - 55.0

   (k) 4.5 x .45

   (l) -8.65 - 7.15

2. Compute the following quotients:

   (a) $\frac{4.08}{2.4}$ (b) $\frac{40.8}{24}$ (c) $\frac{.408}{.24}$ (d) $\frac{408}{240}$

3. Explain why all the quotients in Exercise 2 are the same.

4. Compute the following quotients, correct to two decimal places. (See Example 2 of Section 13.3.)

(a) $\frac{40.8}{2.6}$

(b) $312.48 \div 48.4$

(c) $\frac{580}{3.2}$

(d) $\frac{.05}{3.2}$

(e) $\frac{.005}{.32}$

(f) $875.42 \div .17$

5. During one month, Mr. Sales makes the following deposits in his bank:

$42.50, $97.28, $10.12, $106.77.

What is the total of these deposits?

6. At the beginning of the month, Miss Lane's bank balance was $412.65. During the month she wrote checks for the following amounts:

$5.79, $36.48, $10.20, $75.00, and $85.80

Also, during the month, she made one deposit of $85.80. What was her bank balance at the end of the month?

7. (a) Find the quotient $\frac{3}{4} \div \frac{5}{8}$.

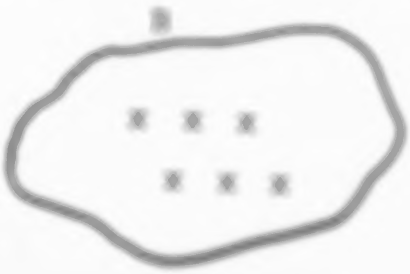
   (b) Find the same quotient as in part (a) by first expressing each number by a decimal fraction.

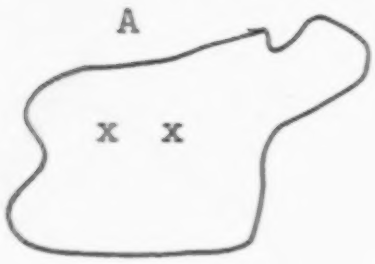
8. If the length of segment X is 3.75 inches, what is the length of segment V, if V = (1.8)X?

9. If a certain material sells for $.45 a yard, how many yards can be bought for $5.40?

13.5 Ratio and Proportion

At the right are two sets of elements, A and B. The number of elements in set A is 2, and the number of elements in set B is 6





We could say, by subtraction, that the number of elements in b is 4 more than the number of elements in A. There is another common way of comparing the sizes of the two sets; this is by stating that the number of elements in B is three times the number of elements in A. That is, $3 \cdot 2 = 6$; or, what amounts to the same thing,

$$\frac{6}{2} = 3.$$

Here we have used the quotient $\frac{6}{2}$ to compare the sizes of the two sets. When used in this way, a quotient is called a ratio. The equation above may be read:

The ratio of 6 to 2 is 3.

There is another way to write $\frac{6}{2} = 3$ when you mean a ratio. It is:

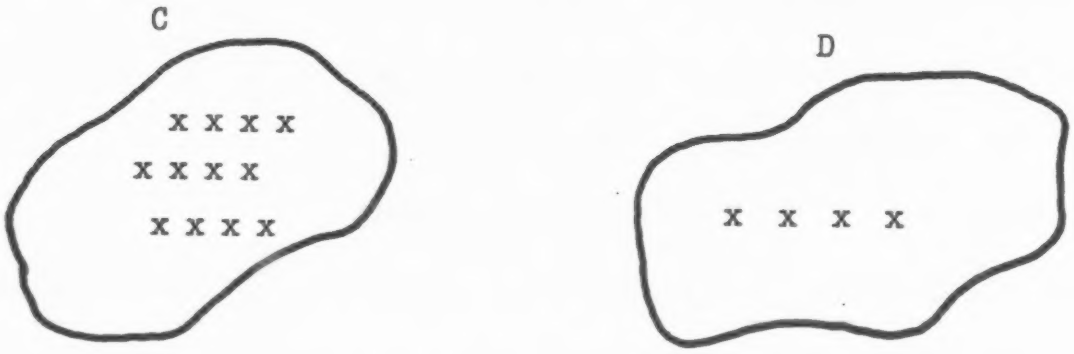
$$6:2 = 3.$$

Notice that we may say:

The ratio of B to A is 3.

This means that if the number of elements in A is multiplied by 3, you get the number of elements in B.

Pictured on the next page are two more sets, C and D, which have 12 elements and 4 elements respectively. What is the ratio of the number of elements in C to the number of elements in D?



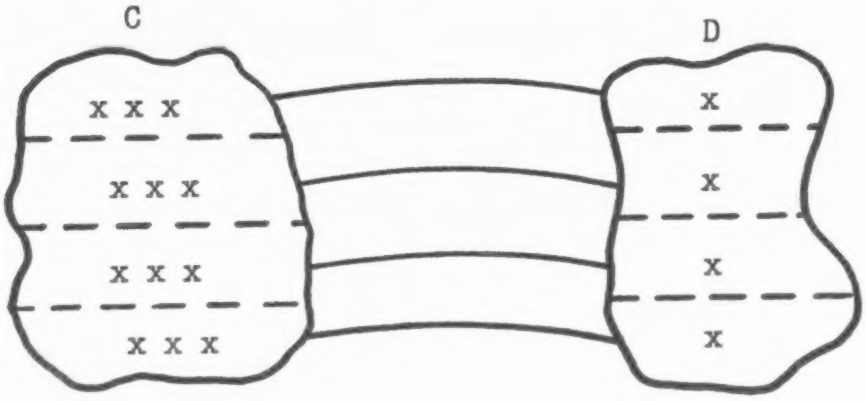
The ratio is $\frac{12}{4}$ (or 12:4); and since $\frac{12}{4} = 3$, there are 3 times as many elements in C as in D. Or, if the number of elements in D is multiplied by 3, the result is the number of elements in C.

Notice that in the two examples above, the ratios (quotients) are equal. That is, $\frac{6}{2} = \frac{12}{4} = 3$. This is true even though the sizes of the sets in the two examples are not the same A sentence such as

$$\frac{6}{2} = \frac{12}{4},$$

which shows that two ratios are equal, is called a proportion. The sentence is sometimes written as "6:2 = 12:4." In this example, we see that $6.4 = 2\cdot 12$. And, in general, two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $ad = bc$. Hence, the test for equal ratios is the same as the test for equivalent fractions which was given in Chapter 12, Section 4.

In terms of the sets being compared, what does it mean to say that two ratios are equal? In the examples above, it means of course that in each case one set is 3 times as large as the other



The above diagram shows for each element in D, there are 3 elements in C. Thus, the sets C and D compare (by means of a ratio) in the same way as a set having 3 elements and a set having 1 element.

Question: Can you draw a diagram like the one above which shows that for every element in A there are 3 elements in B?

Example 1. In Congress, 80 Senators voted on a certain bill, and it passed by 3:1. How many Senators voted for the bill?

This kind of language is often used, and what it means is that the ratio of the number voting for the bill to the number voting against the bill is 3:1. It does not mean that only 3 Senators voted for the bill, and only 1 against. As a matter of fact, in this case 60 Senators voted "yes" and 20 voted "no." Do you see why? Consider how 80 must be separated into two numbers having the ratio 3:1.

Example 2. Two line segments have been drawn. Segment $\overline{CD}$ has length $\frac{1}{2}$ inch, and segment $\overline{AB}$ has length $2\frac{1}{2}$ inches. How do the two segments compare?

$$\begin{aligned} 2\frac{1}{2} \div \frac{1}{2} &= \frac{5}{2} \div \frac{1}{2} \\ &= \frac{5}{2}\cdot\frac{2}{1} \\ &= 5. \end{aligned}$$

Thus, AB:CD = 5. The length of $\overline{AB}$ is 5 times the length of $\overline{CD}$.

Example 2 illustrates that the use of the word "ratio" is not restricted to the comparison of two whole numbers: we may also speak of the ratio of two rational numbers. In general, we say:

> The ratio of a rational number c to a rational number d, $d \neq 0$, is the quotient $\frac{c}{d}$ which may also be written c:d.

Example 3. Let g be the number of girls in a seventh grade class, and let b be the number of boys. If $g = 12$ and $b = 16$, what is the ratio g:b?

$$g:b = \frac{g}{b} = \frac{12}{16} = \frac{3}{4}.$$

The two sets compare in the same way as two sets having 3 and 4 elements. For every 3 girls, there are 4 boys.

Notice also that $\frac{3}{4} \cdot 16 = 12$.

Example 4. Using the numbers from Example 3, what is the ratio b:g?

$$\frac{b}{g} = \frac{16}{12} = \frac{4}{3} \qquad \frac{4}{3} \cdot 12 = 16$$

From all of the examples thus far, the following generalization should be clear:

Example 5 Segment $\overline{AB}$ has a length of 24 inches, and segment $\overline{CD}$ has a length of 8 feet. What is the ratio AB:CD?

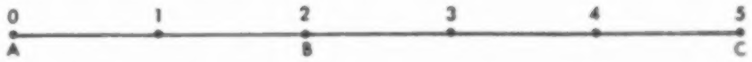
Be careful! It is tempting to say that the ratio is $\frac{24}{8} = 3$. But this is false. Actually the length of $\overline{CD}$ is greater than that of $\overline{AB}$, since 8 feet is certainly longer than 24 inches Since the length of $\overline{CD}$ is measured in feet, we must also express the measurement of $\overline{AB}$ in feet: the length of $\overline{AB}$ is 2 feet. Then the ratio AB:CD is

$$\frac{2}{8} = \frac{1}{4}$$

The length of $\overline{AB}$ is $\frac{1}{4}$ of the length of $\overline{CD}$. We could also change each measure to inches. Then the ratio is 24:96 or 1:4. In comparing quantities of the same kind by ratio, both must be expressed in the same measure.

13.6 Exercises

1. In the drawing below, two segments, $\overline{AB}$ and $\overline{AC}$, have been marked.



   (a) What is the ratio of AB:AC?

   (b) For what dilation D_a would the image of segment $\overline{AC}$ be segment $\overline{AB}$?

   (c) What is the ratio AC:AB?

   (d) For what dilation D_b would the image of segment $\overline{AB}$ be segment $\overline{AC}$?

   (e) If r_1 is the ratio AB:AC, and r_2 is the ratio AC:AB, what is the product $r_1 \cdot r_2$?

2. Find the ratio of the length of U to the length of V if:

   (a) the measure of U is 10 inches; the measure of V is 5 inches.

   (b) the measure of U is 5 inches; the measure of V is 10 inches.

   (c) the measure of U is 3 yards; the measure of V is 18 inches.

   (d) the measure of U is 1 mile; the measure of V is 2000 feet.

   (e) the measure of U is $3\frac{1}{4}$ inches; the measure of V is $1\frac{3}{4}$ inches.

   (f) the measure of U is $1\frac{3}{4}$ inches; the measure of V is $3\frac{1}{4}$ inches.

(g) the measure of U is 2a inches; the measure of V is a inches. ($a \neq 0$)

3. Let a be the number of questions on a test. Let b be the number of questions a student answered correctly. Let c be the number of questions answered incorrectly. If $a = 20$, $b = 17$, and $c = 3$, find the following:
   (a) the ratio of b to a
   (b) the ratio of c to a
   (c) the ratio of b + c to a
   (d) the ratio of b to c
   (e) the ratio of c to b

4. If x and y are two rational numbers such that $x:y = \frac{1}{3}$ give five possible pairs of values for x and y.

5. If c and d are two rational numbers, which number is greater if:

   (a) $c:d = \frac{2}{3}$ (b) $c:d = \frac{3}{2}$ (c) $c:d = 7$ (d) $c:d = 1$?

6. If a and b are two rational numbers such that $\frac{a}{b} = \frac{3}{4}$,
   (a) by what number must you multiply b to get a?
   (b) by what number must you multiply a to get b?

7. Sometimes comparisons are formed in which the numerator and denominator are numbers resulting from measurements involving different units. For example, on a map a scale factor such as "1 inch = 50 miles" means that a segment of 1 inch on the map actually represents a segment of 50 miles in the country-

side. Thus we have the proportional sequence of fractions

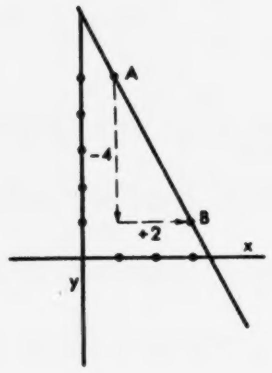
$$\frac{1}{50}, \frac{2}{100}, \frac{3}{150}, \frac{4}{200}, \frac{5}{250} \ldots$$

so that a segment on the map that measures 4 inches, for example, actually represents a segment with measurememnt 200 miles.

(a) On the map described above, a $6\frac{1}{2}$ inch segment represents a segment of what length?

(b) How long a segment must be drawn on the map to represent a 225 mile segment?

8. Thus far we have used only positive numbers in forming ratios. There are problems, however, in which it is sensible to use negative numbers. For example, in the drawing at the right, a line has been drawn in the plane, and two points, A and B, have been marked on the line. The coordinates of B are (3,1). Notice in "moving" from A to B, the x-coordinate increases by 2, which we indicate by +2, and the y-coordinate decreases by 4, which we indicate by -4. Now if we form the ratio



6

$$\frac{\text{change in y-coordinate}}{\text{change in x-coordinate}},$$

we get $\frac{-4}{+2}$, or -2. We say that the slope of the line is -2.

Using this definition of slope, complete the following activities.

(a) Mark the point (3,4), and through this point draw a line whose slope is $\frac{2}{1} = 2$.

(b) Through the point (3,4), draw a line whose slope is $\frac{-2}{1} = -2$.

(c) Mark the point (-2,5), and through this point draw a line whose slope is $\frac{-2}{3}$.

(d) Through the point (-2,5), draw a line whose slope is $\frac{2}{3}$.

(e) Through the point (0,0), draw two lines, one with slope $\frac{4}{5}$ and the other with slope $\frac{-5}{4}$.

(f) Draw two lines, each with slope $\frac{1}{3}$. Draw one line through the point (0,6), and the other through the point (0,2). How do the two lines seem to be related?

13.7 Using Proportions

When we say two segments are in the ratio 2:5 we mean that if the measure of the first segment is 2, the measure of the second segment is 5. We can also write this ratio as the fraction $\frac{2}{5}$. Suppose we desire to have two boards with lengths in the same ratio as the segments. The first board is 4 ft. long. How long must the second board be? If we designate the length by x then the ratio will be $\frac{4}{x}$. But this must be equal to the ratio $\frac{2}{5}$. We thus write

$$\frac{2}{5} = \frac{4}{x}$$

To find the number x, we consider the equality relation of two fractions, namely

$$2 \cdot x = 4 \cdot 5$$

and solving for x find the solution 10. The second board must be 10 ft. long.

Note also, that if any ratio is known, we can form many equal ratios; merely by multiplying the numerator and denominator by the same number. Thus

$$\frac{2}{5} = \frac{2x2}{5x2} = \frac{4}{10}; \quad \frac{2}{5} = \frac{2x\frac{3}{10}}{5x\frac{3}{10}} = \frac{\frac{\cdot 6}{10}}{\frac{15}{10}}$$

$$\frac{2}{5} = \frac{2x\frac{a}{b}}{5x\frac{a}{b}} = \frac{\frac{2a}{b}}{\frac{5a}{b}}$$

Here $\frac{a}{b}$ can be any national number positive or negative but not 0. Why?

Example 1. A picture has measurements 7 inches ("length") and 3 inches ("width"). If the picture is enlarged proportionally so that the new length is 10 inches, what must the new width be?

In all enlargements the ratio of the length to the width must be proportional to the same ratio in the original. The ratios of the lengths to the width are $\frac{7}{3}$ and $\frac{10}{x}$. Thus the proportionality factor $\frac{7}{3}$ must equal $\frac{10}{x}$. Solving $\frac{7}{3} = \frac{10}{x}$, we find $7x = 3 \cdot 10$ or $x = 4\frac{2}{7}$. Therefore, the width of the enlarged picture must be $4\frac{2}{7}$ inches.

Example 2. Solve the proportion

$$\frac{3}{8} = \frac{x}{28}.$$

We solve the proportion as follows:

If $\frac{3}{8} = \frac{x}{28}$ then by the rule of equal fractions $3 \cdot 28 = 8 \cdot x$

$$8 \cdot x = 84$$

$$x = 10\frac{1}{2}$$

In other words, $\frac{3}{8} = \frac{10\frac{1}{2}}{28}$.

If we know the proportionality constant and one member of the missing ratio, we can write the proportion and solve for x.

13.8 Exercises

1. Using whole numbers only,

   (a) write two proportions having the proportionality constant $\frac{5}{6}$;

(b) write two proportions with proportionality constant $\frac{1}{4}$,

(c) write two proportions with proportionality constant .5.

2. In each of the following, find the number x so that the ratios form a proportion.

(a) $\frac{2}{9}$, $\frac{6}{x}$

(b) $\frac{14}{6}$, $\frac{x}{12}$

(c) $\frac{9}{15}$, $\frac{10}{x}$

3. Solve the following proportions.

(a) $\frac{5}{2} = \frac{15}{x}$

(b) $\frac{5}{2} = \frac{12}{x}$

(c) $\frac{3}{7} = \frac{3}{x}$

(d) $\frac{100}{21} = \frac{7}{x}$

(e) $\frac{2}{1} = \frac{9}{x}$

(f) $\frac{1}{2} = \frac{9}{x}$

(g) $\frac{\frac{1}{2}}{3} = \frac{x}{12}$

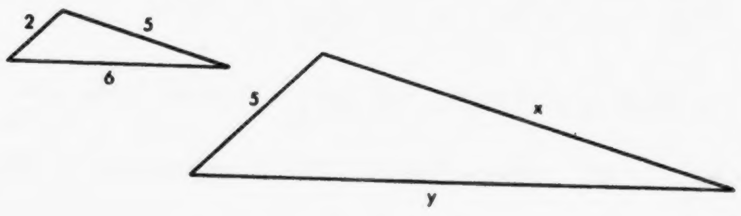
(h) $5:3 = x:15$

(i) $\frac{x}{10} = \frac{a}{2a}$ $(a \neq 0)$

4. The ratio of the number of boys to the number of girls is the same in two different seventh grade classes. In one class, there are 12 boys and 16 girls. In the second class, there are 15 boys. What is the total number of students in the second class?

5. On a certain map there are two segments drawn, one 7 inches long and the second 10 inches long. If the map is enlarged so that the first segment measures 25 inches, how long will the second segment be in the enlargement?

6. Two triangles are drawn below. The triangles are similar, which means that the ratios of the lengths of corresponding sides are all the same. All of the sides in one triangle have their lengths indicated in the figure. In the other triangle, the length of only one side has been marked. Find the lengths, x and y of the other sides.



13.9 Meaning of Percent

In business and social life, one of most common ways of making comparisons is through the use of percent. In the early colonial days this word was written as two words "Per Centum" or "by the hundred." When people borrowed money or goods they paid back in kind by giving e.g. so many dollars for the use of 100 dollars, or so many bushels of corn for the use of 100 bushels. The number 100 was a useful one since 10 was too little for most transactions and 1000 was too much.

Today we think of "percent" as a rational number "one hundredth." The numerical symbol "%" read "percent" is merely another way of writing $\frac{1}{100}$ or .01. If we change a fraction such as $\frac{2}{5}$ for example, to a fraction with denominator 100, in this case $\frac{40}{100}$, we note that $\frac{2}{5}$ is another way of expressing 40%. Thus,

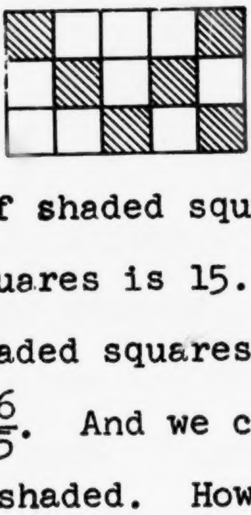
$$\frac{2}{5} = \frac{40}{100} = .40 = 40\% = 40 \cdot \frac{1}{100} .$$

Note that $1 = \frac{100}{100} = 100 \cdot \frac{1}{100} = 100\%.$

Similarly $2 = \frac{200}{100} = 200 \cdot \frac{1}{100} = 200\%, \quad 5 = 500\%,$

$$\frac{3}{2} = \frac{150}{100} = 150\%, \quad \frac{3}{4} = \frac{75}{100} = 75\%.$$

Example 1: In the picture below, there are 15 square regions, and 6 of them have been shaded. What percent of the squares are shaded?



The number of shaded squares is 6; the total number of squares is 15. So the ratio of the number of shaded squares to the total number of squares is $\frac{6}{15}$. And we can say that $\frac{6}{15}$ of the squares are shaded. However, from the discussion above, we know that

$$\frac{6}{15} = 40\%. \quad \text{(Why?)}$$

Therefore, 40% of the squares are shaded.

Example 2: Express $\frac{3}{8}$ as a percent. We must change $\frac{3}{8}$ to a fraction with denominator 100. Hence $\frac{3}{8} = \frac{x}{100}$; $8 \cdot x = 3 \cdot 100$, and $x = 37.5$.

Hence $\frac{3}{8} = \frac{37.5}{100} = 37.5\%$. This is sometimes written as $37\frac{1}{2}\%$.

Example 3: Find the percent equivalent of $\frac{6}{5}$. We use the proportion $\frac{6}{5} = \frac{x}{100}$.

$$6 \cdot 100 = 5 \cdot x$$

$$5 \cdot x = 600$$

$$x = \frac{600}{5} = 120.$$

Therefore, $\frac{6}{5} = 120\%$.

Questions: In a ratio $\frac{a}{b}$ how must a and b be related so that the percent equivalent of the ratio is greater than 100%? less than 100%? equal to 100%?

Example 4: What is the percent equivalent of 3.5?

$$3.5 = 3\frac{5}{10} = 3\frac{50}{100} = \frac{350}{100} = 350\%.$$

Example 5: Express $\frac{1}{2}\% = \frac{1}{2} \times \frac{1}{100} = \frac{1}{200}$.

Question: Which is greater, $\frac{1}{2}$ or $\frac{1}{2}\%$?

Example 5 tells us $\frac{1}{2}\%$ is $\frac{1}{200}$ and surely $\frac{1}{2} > \frac{1}{200}$. Having looked at a number of particular cases, we might consider the general problem of finding the percent equivalent of a ratio. Let $\frac{a}{b}$ be any ratio (of course $b \neq 0$). Then to say that $\frac{a}{b} = x\%$ is to say $\frac{a}{b} = \frac{x}{100}$.

Then we have:

$$b \cdot x = 100 \cdot a$$

$$x = \frac{100a}{b}$$

Otherwise stated, $\frac{a}{b} = \frac{100a}{b}\%$.

13.10 Exercises

1. (a) 50% is the percent equivalent of $\frac{1}{2}$. Write four other ratios for which 50% is the percent equivalent.
   (b) Write five different ratios for which 25% is the percent equivalent.
   (c) Write five different ratios for which 150% is the percent equivalent.
   (d) Write five different ratios for which 100% is the percent equivalent.
   (e) Write five different ratios for which 200% is the percent equivalent.

2. The questions in this exercise refer to the figure below.

A		C		B
	A	C		B
C	C	A		
	C		A	

   (a) What percent of the squares have been marked "A"?
   (b) What percent of the squares have been marked "B"?
   (c) What percent of the squares have been marked "C"?
   (d) What percent of the squares have no mark?
   (e) What is the sum of the percents in questions (a), (b), (c), and (d)?

3. Give the percent equivalent of each of the following:
   (a) .5 (b) .50 (c) .25 (d) 2.5 (e) 1.5
   (f) 1.25 (g) .17 (h) 1.17

4. In the table below, each ratio is to be expressed in the form $\frac{a}{b}$, as a decimal fraction, and as a percent. The first row has been filled in correctly. Fill in all the blanks in the remainder of the table.

Ratio $\frac{a}{b}$	Decimal Fraction	Percent
$\frac{1}{2}$	.50	50%
$\frac{1}{4}$		
	.75	
		20%
	.60	
	.20	
$\frac{1}{8}$		
		87½%
$\frac{4}{5}$		
	.375	
		40%
$\frac{1}{10}$		
		90%
$\frac{1}{1}$		
	.70	
	.05	
$\frac{3}{10}$		
		1%

5. As you recall from Section 12.20, some ratios such as $\frac{1}{3}$ cannot be expressed as terminating decimals, but can be approximated to any desired number of decimal places. How can such a ratio as $\frac{1}{3}$ be expressed as a percent? The question is answered in the same way that all other problems concerning percent equivalents have been answered. Study the steps below:

$$\frac{1}{3} = \frac{x}{100}$$

$$3 \cdot x = 1 \cdot 100$$

$$3 \cdot x = 100$$

$$x = \frac{100}{3} = 33\frac{1}{3}$$

Therefore, the ratio $\frac{1}{3}$ may be expressed as $33\frac{1}{3}\%$. We may also write $\frac{1}{3} \approx 33\%$ where $\approx$ is read "is approximately equal to." Similarly, a better approximation is $\frac{1}{3} \approx 33.3\%$.

Give the percent equivalent of the following ratios:

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{5}{6}$ (d) $\frac{1}{12}$

13.11 Solving Problems with Percents

It is common to see advertisements with statements such as

SALE: 15% OFF ON ALL ITEMS!

Suppose that an item that normally sells for $25.00 is included in the sale advertised above. What should the sale price be? According to the advertisement, 15% of 25.00 should be subtracted from the list price. So the problem is that of finding 15% of 25. Since 15% means $\frac{15}{100}$; and "of" means multiply, we find .15 x 25.00 = 3.75. Since 25.00 - 3.75 = 21.25, the item should sell for $21.25 during the sale.

In the following examples, we solve some other problems, by use of percents.

Example 1. On a test having 20 questions, a student answered 16 of them correctly. What percent of the questions did he answer correctly? That is, what should his percent score be? The ratio of the number of questions answered correctly to the total number of questions is $\frac{16}{20}$;

$$\frac{16}{20} = \frac{80}{100} = 80\%.$$

80% of the questions were correctly answered.

Example 2. On the same test of 20 questions, another student missed 3. What is his percent score?

Since the student missed 3, he answered 17 correctly. The ratio

$$\frac{\text{Number correct}}{\text{total number}} \text{ is } \frac{17}{20} = \frac{85}{100} = 85\%,$$

the student score.

Example 3. In a certain election, 70% of a town's registered voters actually voted. If 3,780 people voted, how many registered voters are in the town?

We know that 70% is .70. We also know that if there are x registered voters $.70 \cdot x$ is the number that voted. Thus $.70 \cdot x = 3780$ or $x = \frac{3780}{.70}$. By division we find $x = 5400$. Check by showing $(.70)(5400) = 3780$.

Example 4. A major league ball player has been at bat 82 times and collected 26 hits. What is his percent of hits? The ratio $\frac{\text{number of hits}}{\text{number of times at bat}}$ is $\frac{26}{82}$ or $\frac{13}{41}$. We find the percent equivalent from:

the proportion

$$\frac{13}{41} = \frac{x}{100}$$

$$1300 = 41x$$

$$31.7 \approx x$$

He has hit approximately 31.7%.

or

by changing $\frac{13}{41}$ to a decimal fraction to the nearest hundredth:

$$\frac{13}{41} = .317$$

$$.32 = 32\%$$

In baseball language this percent expressed to a tenth of a percent ($\frac{1}{10}$ of $\frac{1}{100}$ or $\frac{1}{1000}$) is called the player's "batting average". The player's batting average in this problem is .317.

Example 5. What is $\frac{3}{4}\%$ of 280?

Important! The answer is not 210. (Don't confuse $\frac{3}{4}\%$ with $\frac{3}{4}$.) $\frac{3}{4}\%$ is equal to

$$\frac{3}{4} \cdot \frac{1}{100} = \frac{3}{400}.$$

Then $\frac{3}{400} \times 280 = \frac{840}{400} = 2.10$.

13.12 Exercises

1. Find the following:

   (a) 1% of 500, 5% of 500, $\frac{1}{2}\%$ of 500, $1\frac{1}{2}\%$ of 500, $\frac{1}{2}$ of 500, $\frac{1}{10}\%$ of 500, 10% of 500, 100% of 500.

   (b) 1% of 150, 10% of 150, $\frac{1}{3}\%$ of 150, $1\frac{1}{3}\%$ of 150.

   (c) 1% of 24, 28% of 24, $\frac{3}{4}\%$ of 24, $1\frac{3}{4}\%$ of 24, $\frac{3}{4}$ of 24.

   (d) 1% of 8000, .5% of 8000, 1.5% of 8000, 4.5% of 8000, .5 of 8000.

   (e) 1% of 50, 100% of 50, 200% of 50, 240% of 50,

   (f) 1% of 92, 100% of 92, 300% of 92, 350% of 92.

2. In a high school with 2600 students, 35% of the students are freshmen. How many students are freshmen?

3. In the same high school, there are 390 seniors. What per cent of the school's students are seniors?

4. Suppose the town of Elmwood has a population of 4000 and the town of Springfield has a population of 6000. Complete the following statements.

   (a) The ratio of Elmwood's population to Springfield's population is ______.

   (b) Elmwood's population is ____% of Springfield's population.

   (c) The ratio of Springfield's population to Elmwood's population is _____.

(d) Springfield's population is ____% of Elmwood's population.

5. Complete the statements in the following two columns in the same way the first statement in each column has been completed.

$20 = \frac{20}{40} \cdot 40.$	20 is 50% of 40.
$40 = \_\_ \cdot 20.$	40 is __% of 20.
$20 = \_\_ \cdot 25.$	20 is __% of 25.
$25 = \_\_ \cdot 20.$	25 is __% of 20.
$500 = \_\_ \cdot 400.$	500 is __% of 400.
$400 = \_\_ \cdot 500.$	400 is __% of 500.
$8 = \_\_ \cdot 80.$	8 is __% of 80.
$80 = \_\_ \cdot 8.$	80 is __% of 8.
$16 = \_\_ \cdot 80.$	16 is __% of 80.
$80 = \_\_ \cdot 16.$	80 is __% of 16.
$4.2 = \_\_ \cdot 42.$	4.2 is __% of 42.
$42 = \_\_ \cdot 4.2.$	42 is __% of 4.2.
$1.8 = \_\_ \cdot 180.$	1.8 is __% of 180.
$180 = \_\_ \cdot 1.8.$	180 is __% of 1.8.

6. In a basketball game, a high school team scored 80 points.

(a) If David scored 18 of these points, what per cent of the team's points did he score?

(b) Bill made 27½% of the team's points. How many points did he score?

(c) The number of points David scored is what per cent of the number of points Bill scored?

7. In another game, David made 40% of the team's points. If he made 22 points, how many points did the entire team make?

8. (a) 22 is 40% of what number?
   (b) 80 is 50% of what number?
   (c) 12 is 35% of what number?
   (d) 60 is 150% of what number?
   (e) 7 is 1% of what number?
   (f) 42 is ½% of what number?

9. In a certain state, there is a 4% sales tax. How much sales tax must be paid on purchases of the following amounts?

   (a) $40.00
   (b) $15.00
   (c) $12.50
   (d) $3.25
   (e) $1.00
   (f) $10.00
   (g) $3500.00
   (h) $3499.00
   (i) $9.99

10. Suppose a bank pays 4½% interest per year on savings deposits.
    (a) How much interest should a deposit of $2000 earn in one year?
    (b) How much interest should a deposit of $2000 earn in two years?

11. If the bank in Problem 10 pays interest every six months it will pay only half as much, since 6 months is $\frac{1}{2}$ of a year. (It is the annual interest rate which is 4½%.)

(a) How much will \$1000 earn for six months?

(b) How much will \$2500 earn for six months?

(c) How much will \$2000 earn for three months?

(Hint: 3 months is $\frac{1}{4}$ of a year)

From Exercises 10 and 11, we see that simple interest can be computed from the formula

$$i = p \cdot r \cdot t,$$

where i is the interest, p is the amount of money deposited, r is the rate of annual interest, and t is the time in years.

12. Compute the interest for:

(a) \$500 at 4% for 1 year,

(b) \$500 at 4% for 6 months,

(c) \$500 at 4% for 3 months,

(d) \$1200 at $4\frac{1}{4}\%$ for 1 year,

(e) \$1200 at $4\frac{1}{4}\%$ for 6 months,

(f) \$1200 at $4\frac{1}{4}\%$ for 3 months,

(g) \$1500 at $5\frac{1}{2}\%$ for 2 years,

(h) \$1500 at $5\frac{1}{2}\%$ for $1\frac{1}{2}$ years.

(i) \$750 at 4.2% for 1 year,

(j) \$750 at 4.2% for 6 months.

13. Mr. Smith has kept a deposit of \$1500 in a bank for one year, and the bank pays him \$37.50 interest. What annual rate of interest is the bank paying?

14. Complete the following sentences:

(a) $33\frac{1}{3}\%$ of 3900 is ____.

(b) 20 is ___% of 30.

(c) 30 is ___% of 20.

(d) 20 is 18% of ___.

(e) 20 is 40% of ___.

(f) 108 is 40% of ___.

(g) $2\frac{3}{4}\%$ of 160 is ___.

(h) 2.75% of 160 is ___.

(i) 18 is $66\frac{2}{3}\%$ of ___.

(j) $16\frac{2}{3}\%$ of 66 is ___.

(k) 30 is ___% of 36.

13.13 Presenting Data in Rectangular, Circle, and Bar Graphs

In Chapter 5, a study was made of collecting and representing statistical data in tables and certain graphs. In this section we will construct certain other graphs which give a vivid pictorial summary of the data we wish to present. It might take considerable study to glean the same summary from a table presenting the data in numerical form. Thus popular ways of presentation are the rectangular graph, circle graph, and bar graph.

A seventh grade class made a survey of their junior high school to find the proportional parts of the student body that used various methods of travel to school. When they had gathered the data they first recorded them in a table as shown here.

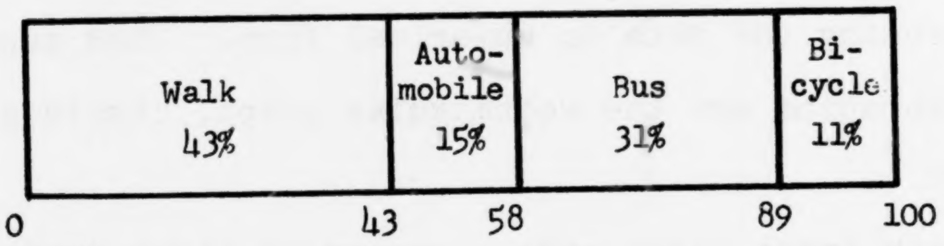
Methods of Transportation to School Used by Students.

Method	Number of Students	Percent of Students
Walk	631	43
Auto	220	15
Bus	455	31
Bicycle	161	11
Total	1467	100

The third column was obtained from the second column by computing the percent each entry was of the total number of students.

To construct a rectangular graph, the length of a rectangle was divided into 100 parts (a good length is 10cm or 100mm). The rectangle was subdivided into rectangular sections at 43, 58, and 89 parts from the left, and marked as shown.

Methods of Transportation to School Used
by Students in a Junior High School



To make a circle graph for the same data, it is necessary to represent 360° as 100%. By proportion we find that one percent,

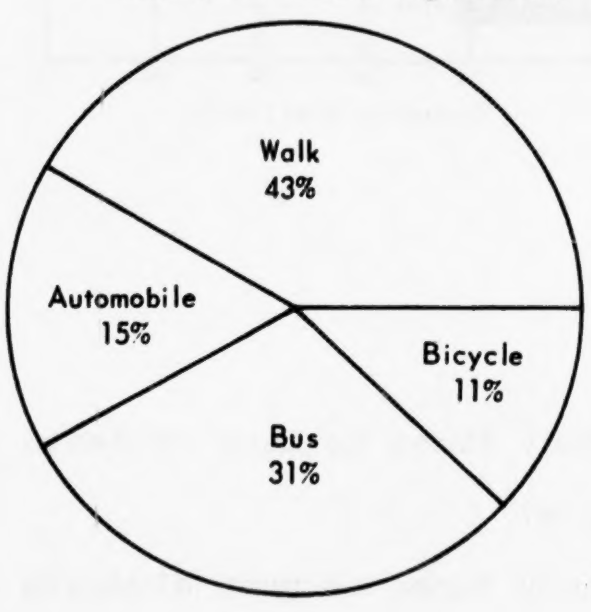
$$\frac{x}{360} = \frac{1}{100} \quad \text{or} \quad x = 3.6.$$

Hence 3.6 degrees represents one percent. The percent column in the table above can now be changed into a degree column, by multiplying by 3.6.

	Percent	Degrees
	43	154.8
	15	54.0
	31	111.6
	11	39.6
Total	100	360.0

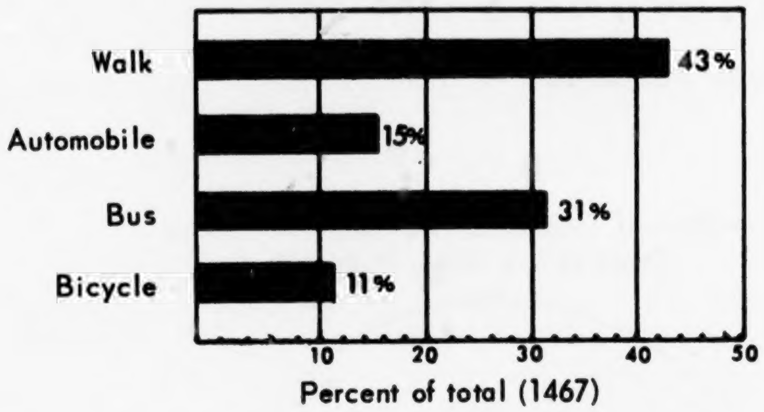
We draw a circle, and with the center of the protractor at the center of the circle, we construct successive angles of approximately 155°, 54°, 111.5° and 39.5°.
The following graph results:

Methods of Transportation to School Used by Students in a Junior High School



To construct a bar graph we draw bars, all the same width, either horizontally or vertically, the length of each bar representing the number in each entry, or the percent in each entry. The space between each bar should be the same width as a bar. In the graph below (a horizontal bar graph), along the horizontal axis a scale of 50 units was selected because the greatest category is 43%. The scale on the axis shows 2% for each division. The bars are then drawn parallel to this scale of length given by the table. Each bar is labelled at the left.

Method of Transportation to School Used by Students in a Junior High School



13.14 Exercises

1. (a) About how many times as many students came by bus as by automobile?

   (b) About how many times as many students walked as came by bicycle?

   (c) What means of travel was used by the smallest number of students? the largest number?

   (d) Which type of graph was the most effective in presenting least and greatest percentages? for comparing the percentages?

2. (a) Obtain information from members of your class on their means of travel to school.

   (b) Gather data from your class on: (1) how many go home for lunch; (2) how many bring their lunch; (3) how many purchase their lunch in the school cafeteria; and (4) how many are in none of the three preceding groups (record this as "other").

3. Present each set of data tabulated in Exercise two by means of graph. Use a rectangular, circle or bar graph.

4. Complete the following table, following the procedure in Section 13.13, and construct a rectangular, circle and bar graph.

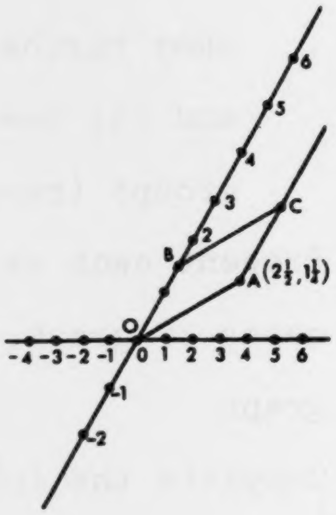
Distribution of Marital Status of Female Workers in a Factory

Status	Frequency	Percent	Degrees
Single	180		
Married	220		
Divorced	25		
Widowed	75		
Total	500		

13.15 Translations and Groups

In preceding chapters we studied translations of a set of points on a line onto itself; of a set of points on one of two parallel lines onto a set of points on the other; of a set of lattice points in a plane onto itself. In this section we extend translations so that they may have as a domain the set of points in a plane whose coordinates, in a given coordinate system, are rational numbers.

Consider the translation, call it t that maps O(0,0) onto A $(2\frac{1}{2}, 1\frac{1}{4})$. What is the image of B $(0, 1\frac{1}{2})$ under t? Name it C. What kind of figure is OACB? Why? The coordinate rule of t is $(x,y) \longrightarrow (x + 2\frac{1}{2}, y + 1\frac{1}{4})$. Is t a one-to-one onto mapping? Why? Does t have an inverse? Let us name it t^{-1}. The -1 denotes an inverse mapping, so t^{-1} is read "the inverse of t" or simply "t inverse." In t^{-1} what is the image of A? of C? of O? The rule for t^{-1} is: $(x,y) \longrightarrow (x - 2\frac{1}{2}, y - 1\frac{1}{4})$



Do you think that every translation of the set of points with rational coordinates in a plane has an inverse? If a translation has rule $(x,y) \longrightarrow (x + a, y + b)$ where x,y,a,b are rational numbers, what is the rule for the inverse of this translation?

Now consider translation t' that maps (x,y) onto $(x + 3\frac{1}{4}, y - \frac{3}{4})$. Under t', what is the image of A $(2\frac{1}{2}, 1\frac{1}{4})$? Is there a single translation that maps O onto this image? What is its rule? Thus, there is a translation which is the composite t'ot, and as you recall, we read it "t' following t."

In particular, what is the composite of t with its inverse t^{-1}? It would seem that it is the identity translation.

In summary, if $(x,y) \xrightarrow{t} (x + a,\ y + b)$,
then $(x,y) \xrightarrow{t^{-1}} (x - a,\ y - b)$.
If $(x,y) \xrightarrow{t'} (x + c,\ y + d)$,
then $(x,y) \xrightarrow{t' \circ t} (x + a + c,\ y + b + d)$,
and $(x,y) \xrightarrow{t^{-1} \circ t} (x,y)$.

You have probably suspected that the set of translations we have been discussing, together with composition, have the properties of a group. Indeed they do, and you are asked to investigate this question further in the following set of exercises.

13.16 Exercises

Assume that all translations in the exercises have for their domain (and range), the set of all points in a plane with rational coordinates in a given coordinate system.

1. Is the composition of two translations an operation? Why?
2. Let T represent the set of all translations and let "o" denote composition of mappings. List the properties that should be proved for (T,o) that will support the claim that (T,o) is a group.
3. Prove that every translation has an inverse in (T,o).
4. Prove that (T,o) contains an identity translation.

5. Prove that (T,o) has the associative property.

6. Prove that (T,o) is a commutative group.

7. Let translation t map (x,y) onto $(x + \frac{1}{3}, y - 2\frac{1}{4})$.

   Find the rule for each of the following:

   (a) tot

   (b) totot

   (c) tototot

   (d) If t is denoted t^1, tot is denoted t^2, totot is denoted t^3, and so on, does the set $\{t^1;t^2,t^3,t^4\ldots\}$ with o form a group? If it does not, explain in what respect it is deficient.

8. Using the data in Exercise 7 find the rule for each of the following:

   (a) t^{-1}

   (b) $t^{-1} \circ t^{-1}$ (denoted t^{-2})

   (c) $t^{-1} \circ t^{-1} \circ t^{-1}$ (denoted t^{-3})

   (d) Does the set $\{t^{-1},t^{-2},t^{-3},\ldots\}$ with o form a group?

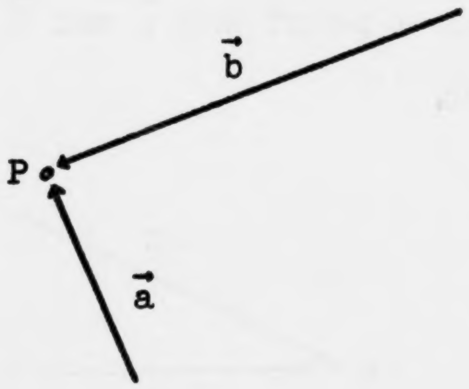
9. Does the set $\{\ldots\ t^{-3},t^{-2},t^{-1},i,t,t^2\ t^3,\ ..\}$ with o form a group, where i is the identity transformation? If not, in what respect is it deficient?

*10. Show that all translations having rules of the form $(x,y) \longrightarrow (x + pa,\ \ y + qb)$, where _a_ and _b_ are fixed rational numbers, and _p_ and _q_ are integers, form a group with the operation "o" (composition of mappings).(Difficult.)

13.17 Applications of Translations

As you might expect, translations have been studied because they are useful in solving certain types of problems. In this section we examine two of these types, both found in science. One problem introduces forces and the other velocities.

We first examine a problem involving forces. Let P, in the diagram below, represent a billiard cue ball which is about to be struck by two billiard cues at the same time. We want to know how the combined effect may be achieved with a single billiard cue.

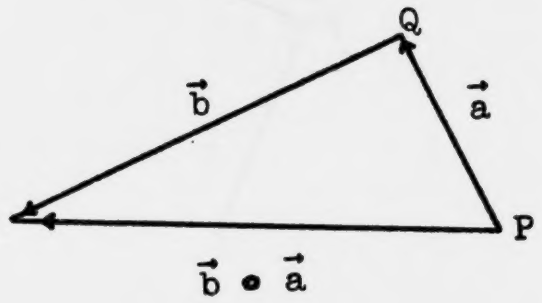


In considering the effect of each cue we must know both the magnitude and the direction of the force which is applied to the ball by the cue. We represent the forces (not the cues) in the diagram by the line segments a and b, together with an arrow at one end of each segment. The length of each segment represents the magnitude of the force. (In our diagram one inch represents a magnitude of 5 pounds.) The line in which the segment lies together with its arrow, indicates the direction of the force. Thus, one force is represented by line segment a directed, at P, as indicated. We denote this force by $\vec{a}$.

The other force, with direction and magnitude as indicated, is represented by line segment b. We denote this force by $\vec{b}$. Since the length of a is one inch, a has a magnitude of 5 pounds. Line segment b is 2 inches long so that the magnitude of b is 10 pounds.

We see, then, that a force is determined by a magnitude and a direction. A translation is determined in the same way. For this reason we might expect to be able to use translations to solve our problem. Our expectations are realized, for "adding" forces is done by composing translations.

Now let us "add" the two forces $\vec{a}$ and $\vec{b}$ described above. To do this we think of P as a point and $\vec{a}$ and $\vec{b}$ as translations. Then we see, in the diagram at the right, that



$P \xrightarrow{\vec{a}} Q$

$Q \xrightarrow{\vec{b}} R.$

Hence $P \xrightarrow{\vec{b} \circ \vec{a}} R.$

$\vec{b} \circ \vec{a}$ is the translation that corresponds to the "sum" of forces. That is, the effect of $\vec{a}$ and $\vec{b}$ together will be to exert a force with a magnitude represented by PR in the line of $\overline{PR}$ and in the direction from P to R. This force is called the resultant of forces $\vec{a}$ and $\vec{b}$. Going back to our original problem, we see that to achieve the same effect with a single cue the cue ball would have to be struck with a force of $11\frac{1}{4}$ pounds. Also, the cue would be sighted along $\overleftrightarrow{PR}$ in the direction from P to R.

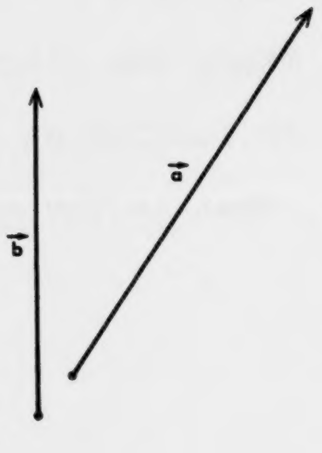
Question: Does $\vec{a} \circ \vec{b} = \vec{b} \circ \vec{a}$? Why or why not?

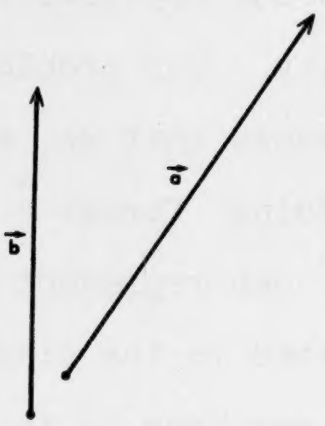
The second application of translations is to problems involving velocity. Our problem will then be to "add" velocities in the same sense that we "add" forces. We can reinterpret our problem of "adding" forces $\vec{a}$ and $\vec{b}$ by thinking of them as velocities. Then $\vec{a}$ can represent a speed of 5 miles per hour in the direction indicated in the diagram, and $\vec{b}$ can represent a speed of 10 miles per hour in the direction indicated in the diagram. Here again the lengths of $\vec{a}$ and $\vec{b}$ represent the magnitudes (speeds in miles per hour) of the velocity, and the line of the segment, with an arrow, represents the direction. Here we might be solving a problem such as the following:

> A toy boat is propelled by its engine with velocity $\vec{b}$. A wind is blowing with velocity $\vec{a}$. In what direction, and with what speed, does the boat actually move? (That is, with what velocity does the boat move?)

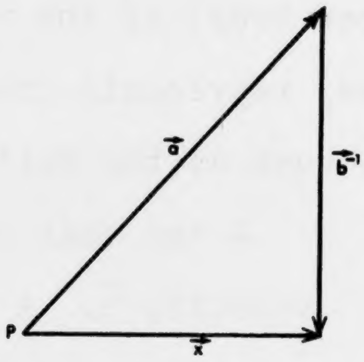
The answer is found in exactly the same manner as "adding" forces. The answer for this problem then, is: the boat moves at the rate of $11\frac{1}{4}$ miles per hour in the direction of $\overline{PR}$ as indicated by its arrow.

We end this section with another example. Suppose a boat actually moves in the direction of $\vec{a}$ (shown at the right) with a speed of 20 miles





per hour, but its propeller and engine operate to make it move in the direction of $\vec{b}$ (shown at right) with a speed of 15 miles per hour. The difference is due to the wind. In what direction is the wind blowing and with what speed? Note that $\vec{a}$ is 2 inches long and $\vec{b}$ is $1\frac{1}{2}$ inches long. Then the scale in the drawing is 1" = 10 mi.



To solve this problem think of $\vec{a}$ and $\vec{b}$ as the translations corresponding to the velocities and $\vec{x}$ as the translation corresponding to the velocity of the wind. Since $\vec{a}$ is the composite of $\vec{b}$ with $\vec{x}$ we have: $\vec{b} \circ \vec{x} = \vec{a}$.

We solve for $\vec{x}$ and find $\vec{x} = \vec{b}^{-1} \circ \vec{a}$. This guides us in solving the problem. Study the diagram and be able to explain how it was made. In looking at the diagram, start at P. How long is segment x? What is the speed of the wind?

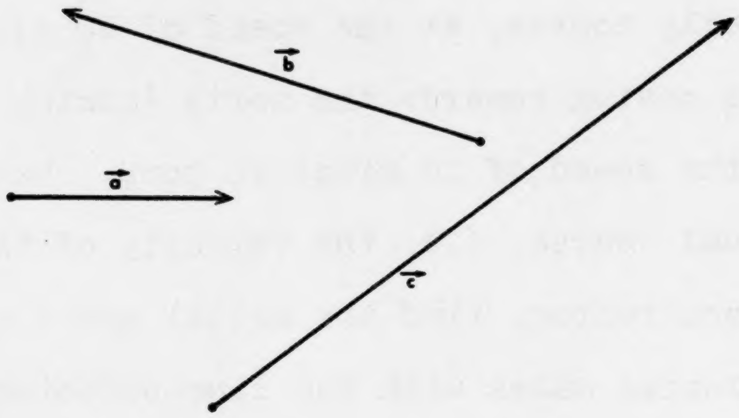
13.18 Exercises

1. The propeller and engines of a ship are set to propel it on an easterly course, at the speed of 20 miles per hour. The wind is moving towards the north (coming from the south) at the speed of 10 miles an hour. Make a diagram of the actual course, i.e. the velocity of the ship. Using ruler and protractor, find the actual speed and find what angle the course makes with the line pointing to the north. (Use the scale: 1 inch = 10 miles.)

   Note: We neglect the force of the flow of the water, called "drift."

2. Answer the same questions asked in Exercise 1 for each of the following cases.
   (a) Intended course of ship is northeast; speed is 15 miles per hour; the wind comes from the west at 30 miles per hour. (Use the scale: 1 inch = 10 miles.)
   (b) Intended course is northwest; speed is 18 miles per hour; the wind comes from the southwest at 24 miles per hour. (Use the scale: 1 inch = 6 miles.)
   (c) The ship's intended course is southeast; speed 15 miles per hour; the wind comes from the northwest, at 5 miles per hour. (Do you need a diagram for this problem?)

In Exercises 3 - 7 use the segments shown below to represent forces. The scale we used to draw them is 1 inch = 10 pounds.



3. Suppose forces $\vec{a}$ and $\vec{b}$ are applied to an object. Use a diagram to find the resultant and compute the magnitude (number of pounds) of the resultant force.

4. Proceed as in Exercise 3 given:
   (a) forces $\vec{a}$ and $\vec{c}$ are applied together.
   (b) forces $\vec{b}$ and $\vec{c}$ are applied together.
   (c) $\vec{a}, \vec{b}$, and $\vec{c}$ are applied together.

5. Suppose force $\vec{a}$ is applied and $\vec{c}$ is the resultant. Find the force $\vec{x}$ that was applied together with $\vec{a}$, and compute its magnitude.

6. Suppose force $\vec{b}$ is applied and $\vec{c}$ is the resultant. Find the force $\vec{x}$ that was applied together with $\vec{b}$, and compute its magnitude.

7. Suppose $\vec{c}$ is applied and $\vec{a}$ is the resultant. Find the force $\vec{x}$ that was applied together with $\vec{c}$ and compute its magnitude.

8. Suppose two forces are applied and the resultant leaves the object in its original position. What must have been true of the two forces?

13.19 Summary

1. If x is any rational number, then D_x is a dilation which maps each point into a point x times as far from the origin.

2. Decimal fractions may be used in finding sums, differences, products, and quotients of rational numbers.

3. Two sets may be compared by means of a ratio. The ratio of a number x to a number y is the quotient $\frac{x}{y}$, also written as $x:y$. (It is understood that $y \neq 0$.)
   If $\frac{x}{y} = r$, then $x = r \cdot y$.

4. If two ratios, $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$, are related so that

   $$\frac{a_1}{b_1} = \frac{a_2}{b_2},$$

   then the ratios are said to be in proportion. An equation of the form $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ is called a proportion.

5. The ratio $\frac{a}{100}$ is also written as "a%" and read "a percent." Every ratio can be expressed in the form $\frac{a}{b}$, where a and b are integers, or as a decimal fraction, or as a percent. Many mathematical problems occuring in everyday life are expressed in the language of percents.

6. If T is the set of all translations of form $(x,y) \xrightarrow{t} (x + a,\ y + b)$, where a and b are rational numbers, and if $\circ$ is composition of translations, then $(T, \circ)$ is a commutative group.

13.20 Review Exercises

1. (a) What is $\frac{2}{9}$ of 18?
   (b) What is 15% of 200?
   (c) What is .35 x 650?

2. If a 12% tax must be paid on $3500, how much tax must be paid?

3. During a sale, a store reduces all prices by 20%. What is the sale price of a television set which normally sells for $220.00?

4. In a school, 35 of the 225 boys go out for basketball. What percent of the boys in the school go out for basketball?

5. 4% of the girls in the school are cheerleaders, and there are 8 girl cheerleaders. How many girls are there in the school?

6. A bank pays interest at an annual rate of $4\frac{3}{4}\%$. How much will $4000 earn during a 6-month period?

7. Compute the following:
   (a) $8.875 + 44.327$
   (b) $102.54 - 87.39$
   (c) $21.8 - 39.3$
   (d) $(2.3) \times (4.3 \times 7.5)$
   (e) 5.6×8.75
   (f) $\frac{6.138}{4.65}$
   (g) $\frac{6.138}{1.32}$

8. In a certain city there are 4200 Democrats and 3600 Republicans. What is the ratio of Democrats to Republicans? (Express the answer as an irreducible fraction.) Then fill in the following blanks so that a true statement results:

   For every ____ Republicans, there are ____ Democrats.

9. In a student council, there are 24 members. With all members voting, Jim won the presidency by a 3:1 vote. How many voted for Jim?

10. Solve the following proportions:

(a) $\frac{5}{3} = \frac{35}{x}$ (b) $\frac{2}{7} = \frac{9}{x}$ (c) $\frac{2}{7} = \frac{x}{9}$ (d) $\frac{9}{x} = \frac{x}{4}$

11. Write the coordinates of the image of each of the following points under the dilation

$$D_{-\frac{5}{3}}$$

A $(\frac{3}{5}, \frac{3}{5})$, B $(-\frac{3}{5}, -\frac{3}{5})$, C (2, 4), D (0,9),

E (9,0), F (-1, 1).

12. Let t be the translation in Q X Q which has the following rule:

$$(x,y) \xrightarrow{t} (x + \frac{5}{3}, y - \frac{2}{5})$$

(a) What is the rule for tot?

(b) What is the rule for t^3 ?

(c) What is the rule for t^{-1} (the inverse of t)?

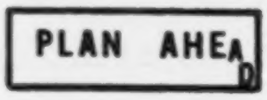
(d) What is the rule for t^{-2} ?

CHAPTER 14

ALGORITHMS AND THEIR GRAPHS

14.1 Planning a Mathematical Process

Many of you have seen the humorous sign



The humor, of course, is in the fact that the painter did not heed the advice he was giving to others. To avoid crowding the letters on future signs, the novice sign painter could request that his supervisor provide detailed instructions for painting the words PLAN AHEAD on a piece of cardboard of a given size. The instructions probably would be something like this:

Use a ruler to find the length of the board.

Count the number, n, of letters and blank spaces needed to print the message.

Divide the length of the board by the number of spaces needed to find the length of each space.

Mark off n spaces on the board -- each of the required length.

Paint the letters in the appropriate spaces.

If one blank space is to precede the "P," one is to be between the two words, and one is to follow the "D," what is the number of letters and blanks to be for the sign? If the length of the sign is to be 12 inches, how long will each space be?

Are the instructions clear and complete? If not, what modifications should be made?

Lists of instructions such as those used by the sign painter occur frequently. The instruction manuals that come with almost any toy or machine, and the recipes your mother clips from newspapers and magazines are examples of such lists. Instruction lists occur also in mathematics. For example, you are familiar with the following instructions for averaging a set of test grades:

Add all grades in the list and obtain the sum S.

Count the number, n, of grades in the list.

Divide S by n to determine the average of the n grades.

A list of instructions or a recipe is useful only if the process described finally comes to an end. Otherwise, no sign would ever be finished, no dish prepared for the table and no average recorded on your report card. In mathematics, a list of instructions describing a process which eventually comes to an end is called an algorithm or algorism after the Latin name of the Arab mathematician Mohammed al Khowarizmi who collected many algebraic recipes in a book entitled ilm al-jabr wa'l muqabelah (c. 800).

Algorithms occur in all areas and at every level of mathemetics. You already know and use a number of algorithms. The algorithm for adding two two-digit numbers and the algorithm for dividing one number by another are familiar to you. Algorithms are especially useful in work with electronic computers. Although computers can perform mathematical operations at high speeds, they must first receive a detailed sequence of instruction; that is, the computer must be given an algorithm.

Algorithms may be written out as in the preceding example, recorded on magnetic tape, or represented in a variety of other ways. One procedure for recording the sign painter's algorithm would be to write the instructions on individual index cards. If the cards are placed in a pocket or drawer and reassembled at a later time, trouble may develop. Suppose the sign painter reassembled the cards as in Figure 14.1.

Count the letters and blanks needed.

Paint the letters.

Measure the signboard.

Divide the length of the signboard
by the number of spaces needed.

Mark off the necessary number of
spaces on the signboard.

Figure 14.1

What sort of a sign might be produced if these instructions were followed? Do you see that not only are the instructions important, but their arrangement is important as well?

We can avoid this difficulty either by numbering the cards or by indicating with arrows the sequence in which the steps are to be performed, as in Figure 14.2.

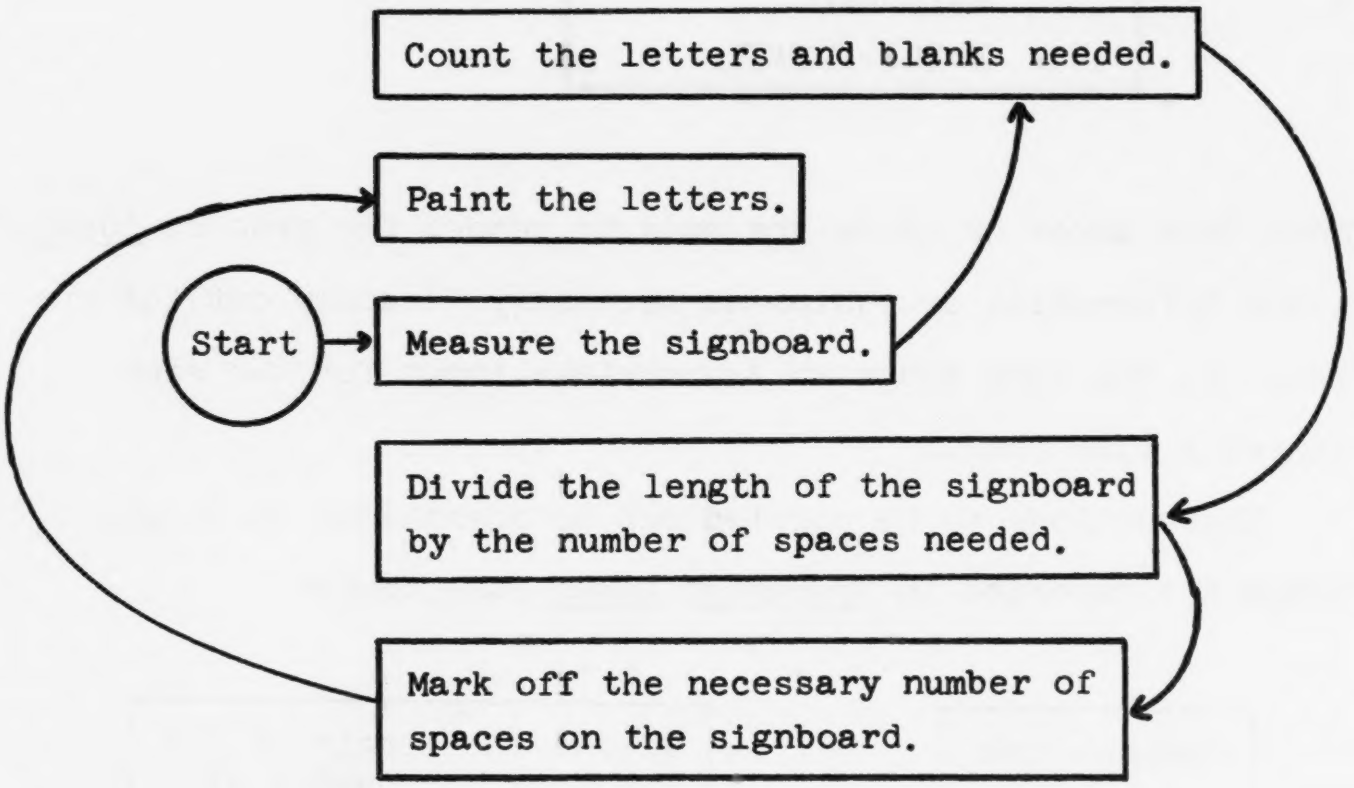
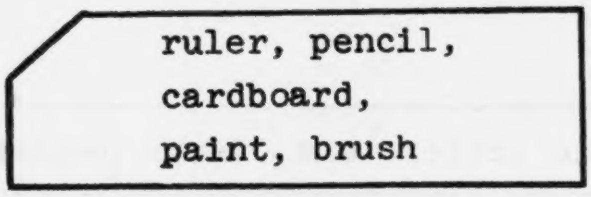

Count the letters and blanks needed.
Paint the letters.
Start
Measure the signboard.
Divide the length of the signboard by the number of spaces needed.
Mark off the necessary number of spaces on the signboard.

Figure 14.2

The picture of the sign painter's algorithm with separate instructions recorded in boxes and the boxes joined by arrows is, in a sense, a diagram or a graph of the algorithm. Graphs of algorithms are called flow charts. Two things should be apparent in every flow chart -- the instructions themselves, and the "flow" or sequence in which they are to be performed.

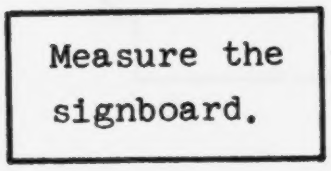
Since flow charts are useful for recording algorithms, and since algorithms are essential in machine computation, mathematicians who prepare programs for electronic computers have developed a standard form and language for the construction of flow charts. Facts and equipment important for specific processes are recorded in data boxes like this:



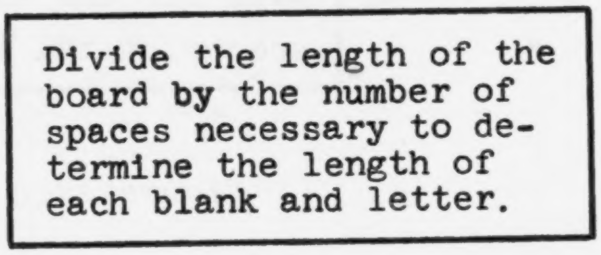
ruler, pencil,
cardboard,
paint, brush

These data boxes or cards are used to record the process input -- the information or equipment necessary to carry out the process. Is the card above an appropriate input for the sign painter's flow chart?

Instructions to be carried out or operations to be performed are recorded in operation boxes like these:

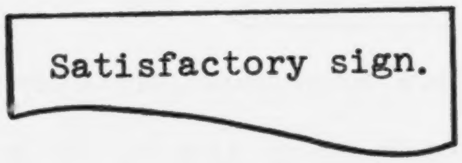


Measure the signboard.



Divide the length of the board by the number of spaces necessary to determine the length of each blank and letter.

Information obtained by means of the process described in the flow chart is recorded in output boxes like this:



Satisfactory sign.

Note that each type of box has a characteristic shape.

Using these conventions, try to complete the flow chart for the sign process in Figure 14.3.

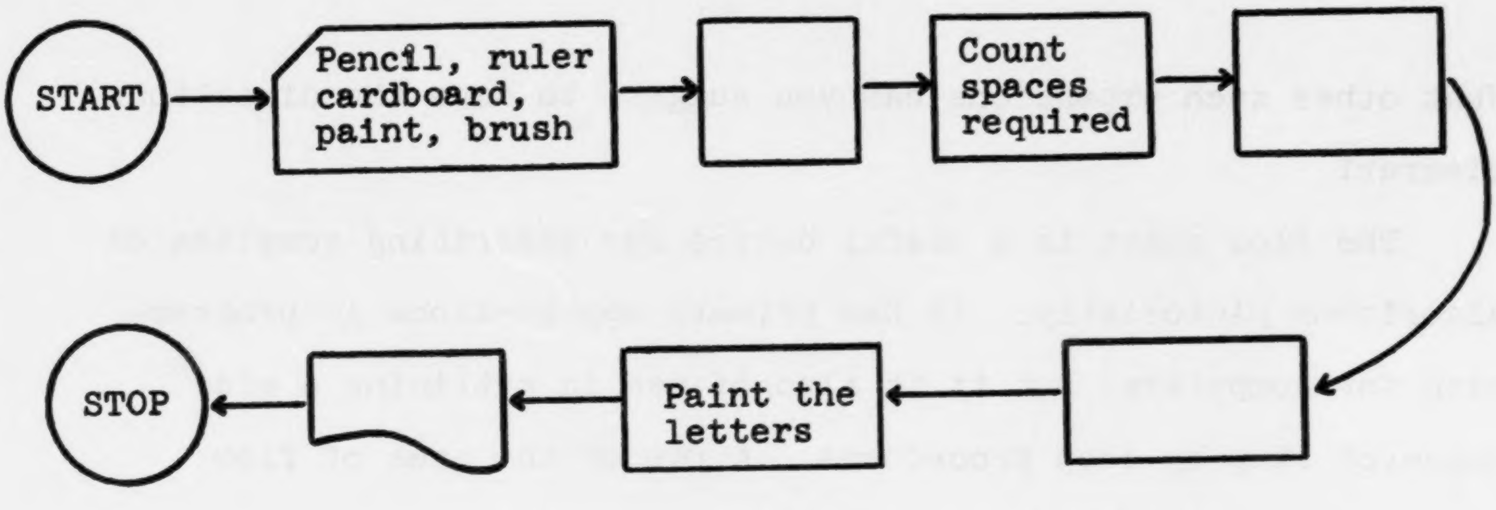

START
Pencil, ruler cardboard, paint, brush
Count spaces required
Paint the letters
STOP

Figure 14.3

While both the content and sequence of the instruction boxes are important, it is often possible to give instructions in several ways, each producing the same result. Is there any rearrangement of the instructions in the PLAN AHEAD flow chart that still would produce an acceptable sign? It is possible also to expand the instructions within one of the boxes, developing a flow chart within a flow chart. For instance, we could replace the box which commands, "Count the number, n, of letters and blanks needed to print the message," by the sequence of three operation boxes shown in Figure 14.4.

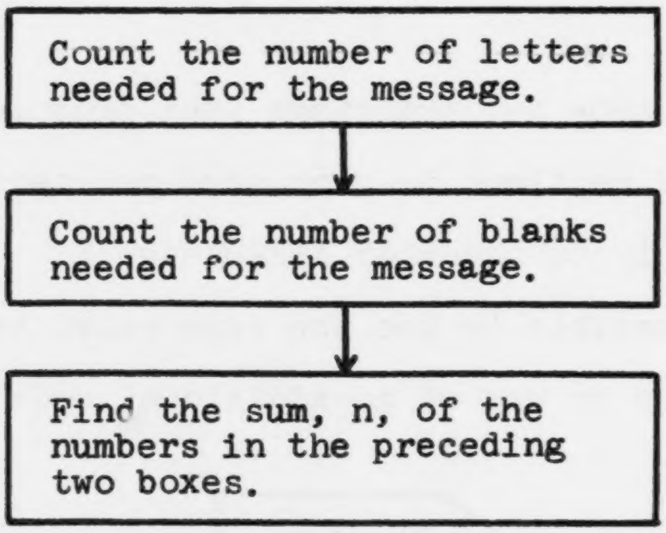

Count the number of letters needed for the message.
Count the number of blanks needed for the message.
Find the sum, n, of the numbers in the preceding two boxes.

Figure 14.4

What other such expansions can you suggest to make the directions clearer?

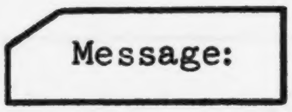
The flow chart is a useful device for describing complicated algorithms pictorially. It has primary applications in programming for computers, but it is also of use in outlining a wide range of step-by-step procedures. A few of the uses of flow charts are illustrated in the following exercises.

14.2 Exercises

1. Try to construct a flow chart of the main steps describing
    - (a) how you got to school today;
    - (b) your daily schedule at school;
    - (c) how to draw a circle with compasses or a string and pencil;
    - (d) how to find the average of two numbers;
    - (e) how to find the factors of a number;
    - (f) a game you play.

   Does the order of the boxes affect the chart you have constructed?

   If possible, show two orderings that produce the same result.
2. (a) Do the directions in your sign painter's flow chart apply only to the PLAN AHEAD sign ?
   - (b) Is it possible to use the same chart to produce different signs by use of an additional data box?



If so, where should this data box be placed in the chart?

(c) What would the message boxes be for painting

(1) CAUTION: RUTABAGAS

(2) U.N.C.L.E.

(3) _________ _____________ (your name)

(d) What number of letters and blanks is needed in each sign suggested in (c)?

(e) If you have a signboard 24 inches long, how long will each letter and blank be in each of the three signs?

3. Classify the following as input, output, or operation boxes:

(a) Add.

(b) Seven.

(c) Apples and Oranges

(d) 2, 3, 7, 11.

(e) Skip school.

(f) A school skipper.

(g) An hour after school for two weeks.

4. If possible, arrange the following cards in order to give a flow chart that makes sense.

(a) Ball Shoot Aim Basket Score

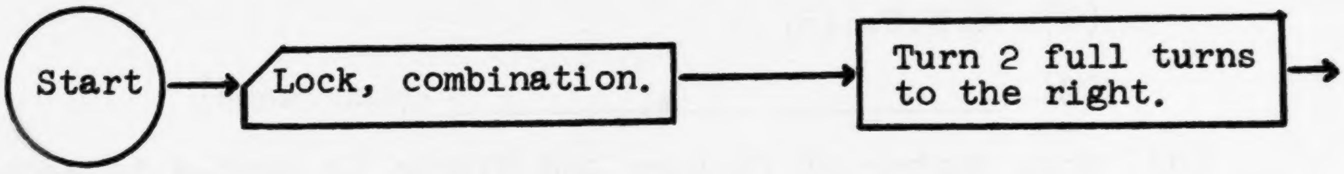
(b) Needle Sew Thread Dress Cut Cloth

(c) Ball Glove Catch Throw Bat Out Hit

5. Try to write a flow chart for multiplying two fractions which can be followed successfully by someone who doesn't know

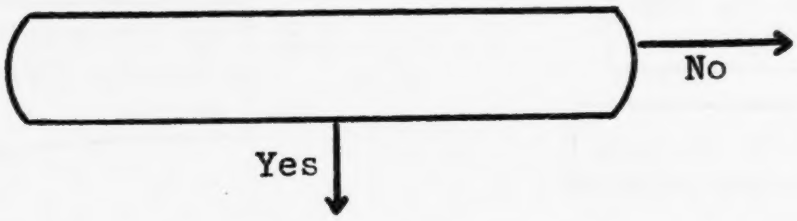
what a fraction is.

6. Write a flow chart for averaging two numbers.
7. Can you write directions for the process of opening a combination lock? You might begin with



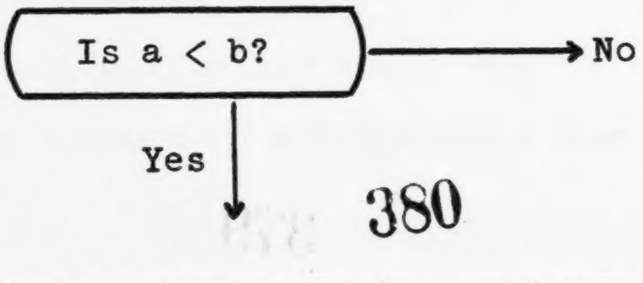
14.3 Flow Charts of Branching Algorithms

In the preceding section you were introduced to flow charts of simple algorithms. The flow charts consisted of input, output, and operation boxes that were arranged in a definite order. Often in flow charts of more complicated algorithms, you will find a fourth kind of box. This new box is called a decision box, and usually looks like this:



Decision boxes contain questions. In flow charts of mathematical processes, decision boxes usually ask whether two quantities are equal or whether a certain inequality holds between them.

For example, the decision box



asks whether the number a is less than b. The outcome of a decision box is "yes" or "no." Often decision boxes are used to create "forks" or branches in a flow chart. The chart may indicate that one set of instructions is to be followed if the answer is "yes," and a different set if the answer is "no."

As an example of a flow chart that includes a decision box, consider the trial and error procedure in Figure 14.5 for painting the PLAN AHEAD sign.

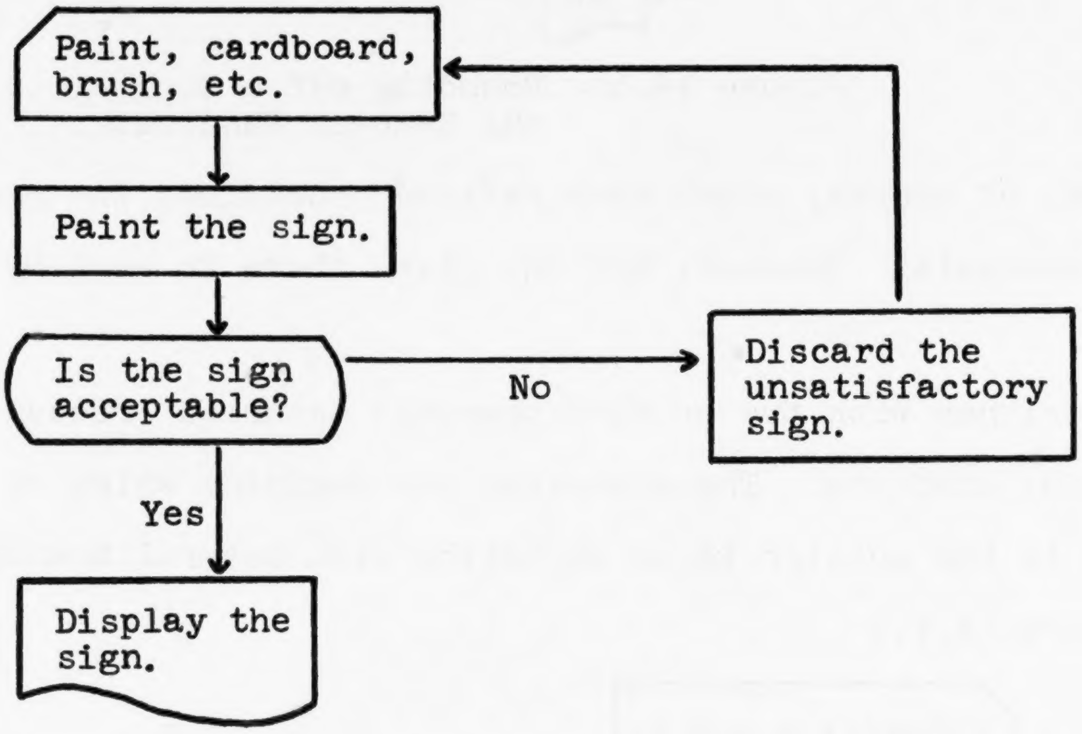

Paint, cardboard, brush, etc.
Paint the sign.
Is the sign acceptable?
No
Discard the unsatisfactory sign.
Yes
Display the sign.

Figure 14.5

Of course, there is no guarantee that the process ever will result in an acceptable sign; hence, the chart is not the flow chart of an algorithm.

For an example of a bona fide algorithm, which includes a decision box, take the usual method of rounding off a decimal to the nearest hundredth. To round off, say, .abcd, where a, b, c, and d are digits, we consider only the thousandths digit c. If $c < 5$, then .abcd rounded to hundredths is simply .ab. If $c \geq 5$, then .abcd rounded to hundredths is .ab + .01. In flow chart form this algorithm is displayed in Figure 14.6.

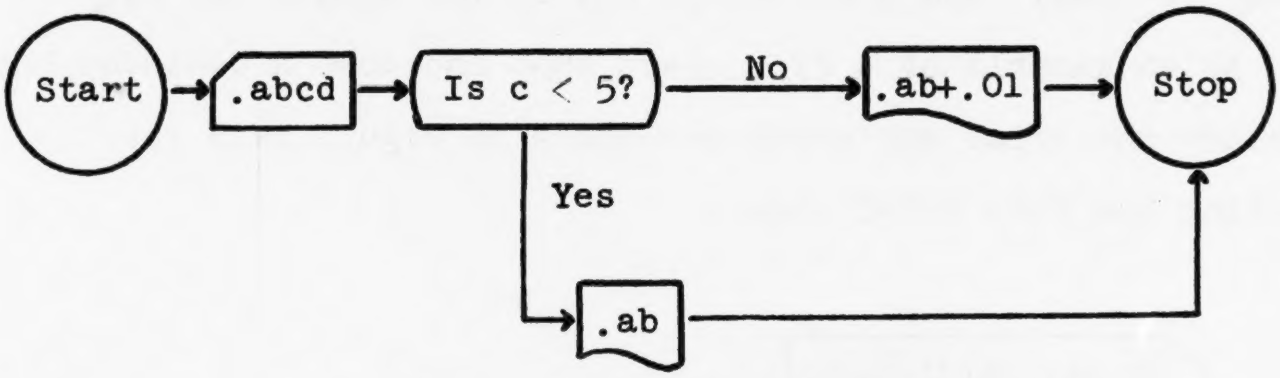

Start
.abcd
Is c < 5?
No
.ab+.01
Stop
Yes
.ab

Figure 14.6: Rounding off a Decimal to the Nearest Hundredth.

There are, of course, other more refined procedures for rounding off decimals. However, the one given above is used by most students.

Algorithms with two or more branches occurred frequently in previous chapters. The algorithm for deciding which of two integers is the greater is an algorithm with several branches. (See Figure 14.7.)

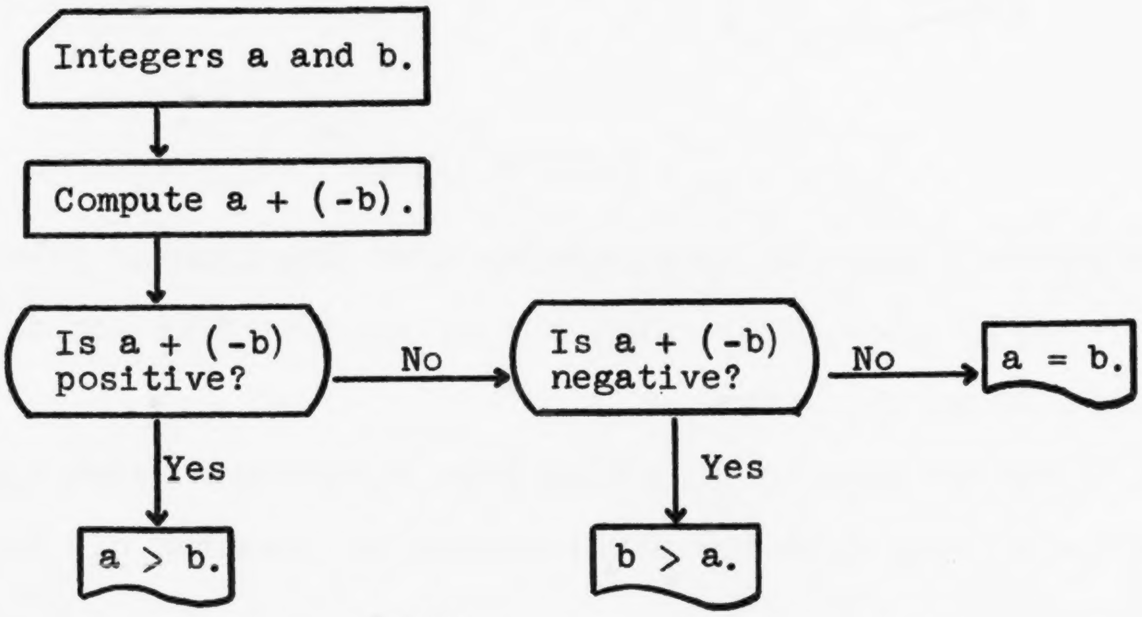

Integers a and b.
Compute a + (-b).
Is a + (-b) positive?
No
Is a + (-b) negative?
No
a = b.
Yes
a > b.
Yes
b > a.

Figure 14.7: Determining if a > b, b > a, or a = b.

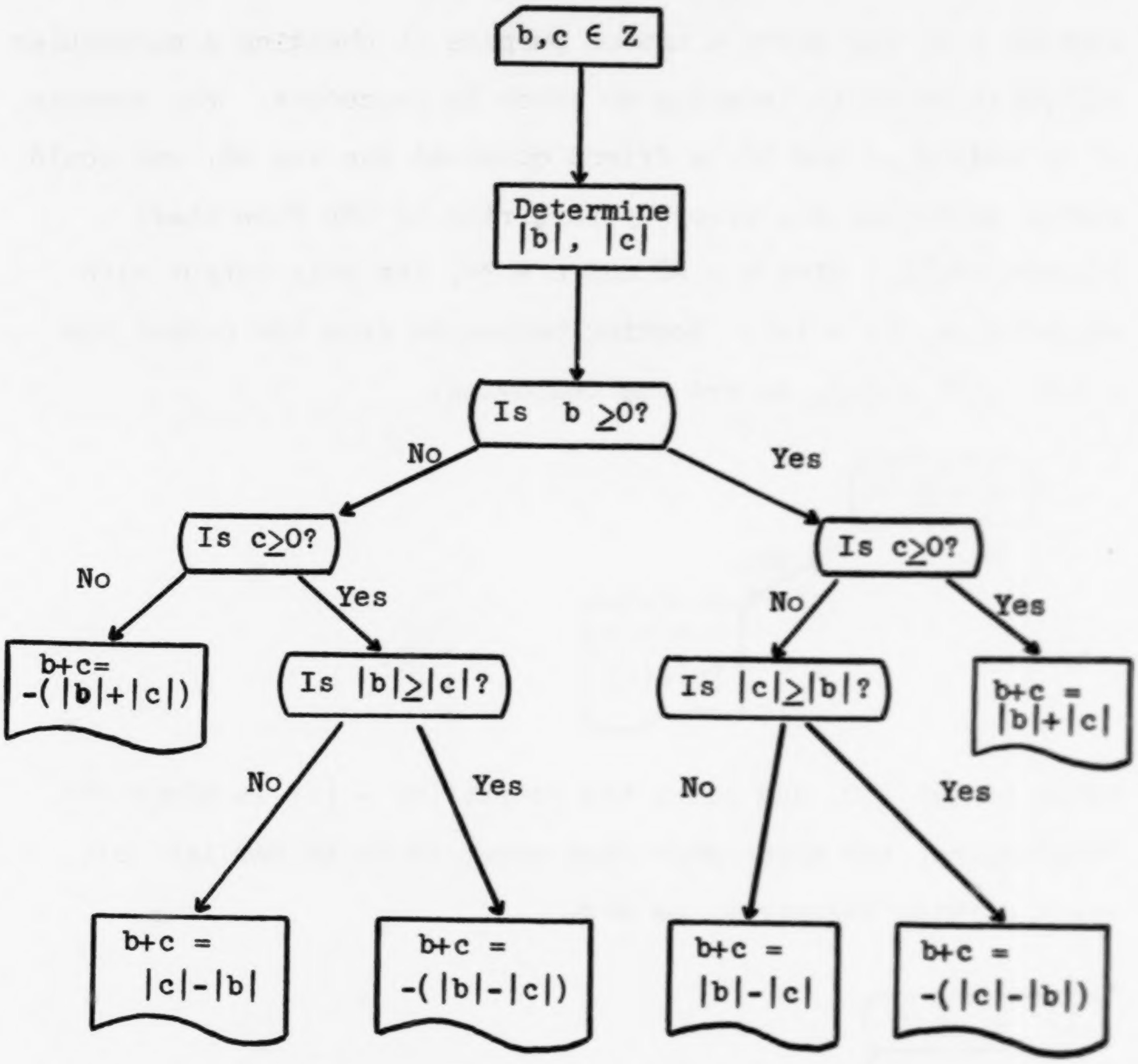
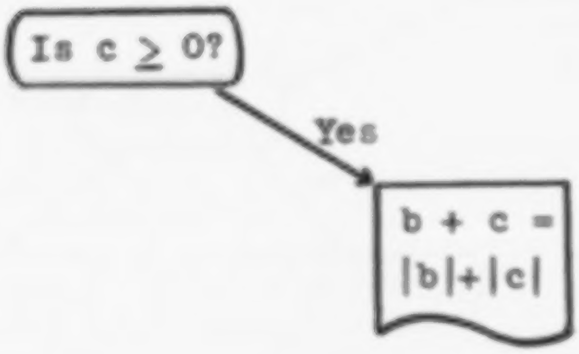
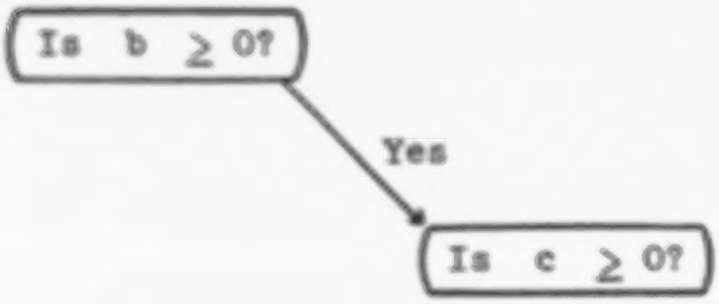

b,c ∈ Z
Determine |b|, |c|
Is b ≥0?
No
Yes
Is c≥0?
No
Yes
Is c≥0?
No
Yes
b+c= -(|b|+|c|)
Is |b|≥|c|?
Is |c|≥|b|?
b+c = |b|+|c|
No
Yes
No
Yes
b+c = |c|-|b|
b+c = -(|b|-|c|)
b+c = |b|-|c|
b+c = -(|c|-|b|)

Figure 14.8: Adding Two Integers.

Similarly, the algorithm for computing the sum of two integers is a multi-branch routine. (See Figure 14.8) While the preceding flow charts may seem to you to be of little practical value, once the algorithms they represent have been learned they can serve a useful purpose in checking a particular computation or in locating an error in procedure. For example, if in adding -8 and 36, a friend obtained the sum 44, you could easily point out his error by referring to the flow chart (Figure 14.8). With b = -8 and c = 36, the only output with value 44 is |b| + |c|. Working backwards from the output box b + c = |b| + |c|, we see the following:



Since c = 36 ≥ 0, and since the output |b| + |c| is along the "Yes" arrow, the error must have occurred in an earlier cell. Again working backwards, we see:



Since b = -8 < 0, the error very probably occurred here.

Once the location and nature of the error are determined, it is a relatively simple matter to repeat the routine avoiding this same mistake.

14.4 Exercises

1. Which of the following might appear in a decision box?

    (a) Is it raining?

    (b) $3 + 2$

    (c) $3 + 2 = a$

    (d) It is raining.

    (e) $a > 0$

    (f) $a + 5$

2. Write a flow chart for computing the product of two integers.

3. Write a branching flow chart for finding the largest of a set of three integers.

4. Write a flow chart for arranging three integers in increasing order.

5. Write a flow chart for arranging three integers in increasing order and removing duplicates, if any.

6. Write the flow chart of an algorithm for finding a single heavy ball in a set of eight balls.

7. Use flow charts to locate the errors in the following computations:

    (a) $(-8) + (-36) = 28$.

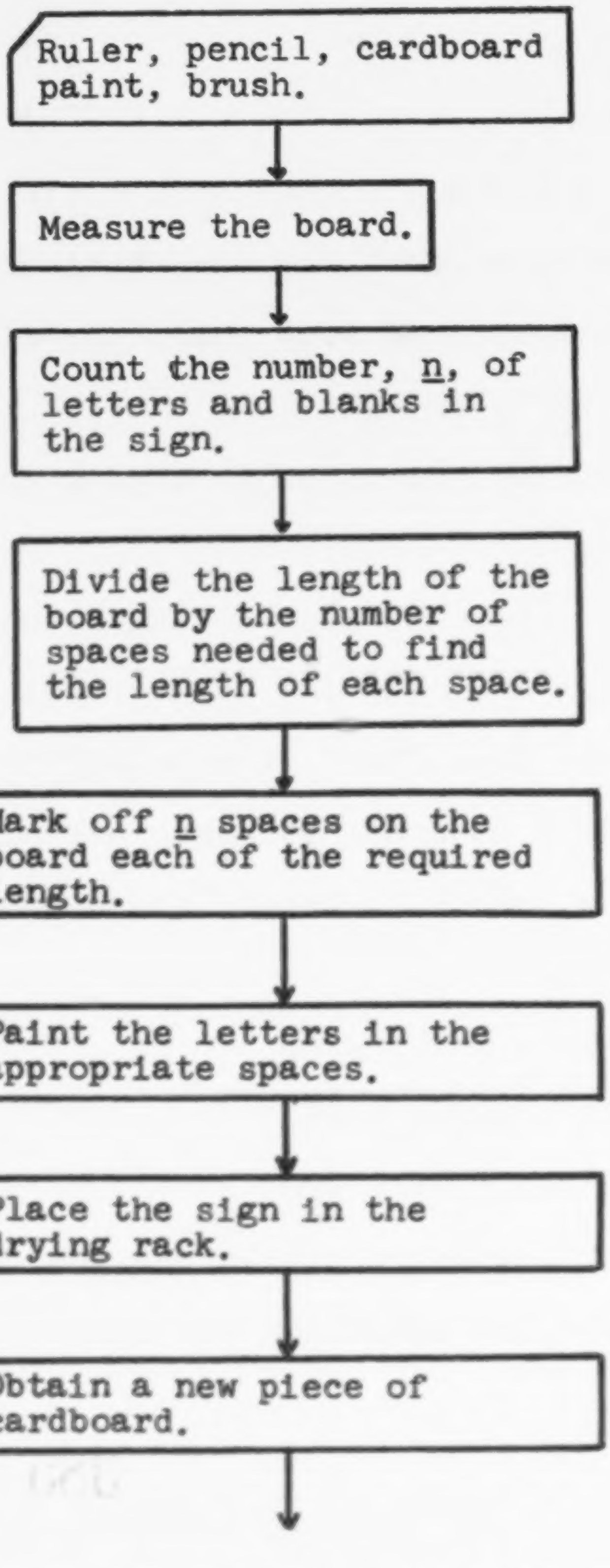
    (b) $(8) + (-36) = 44$.

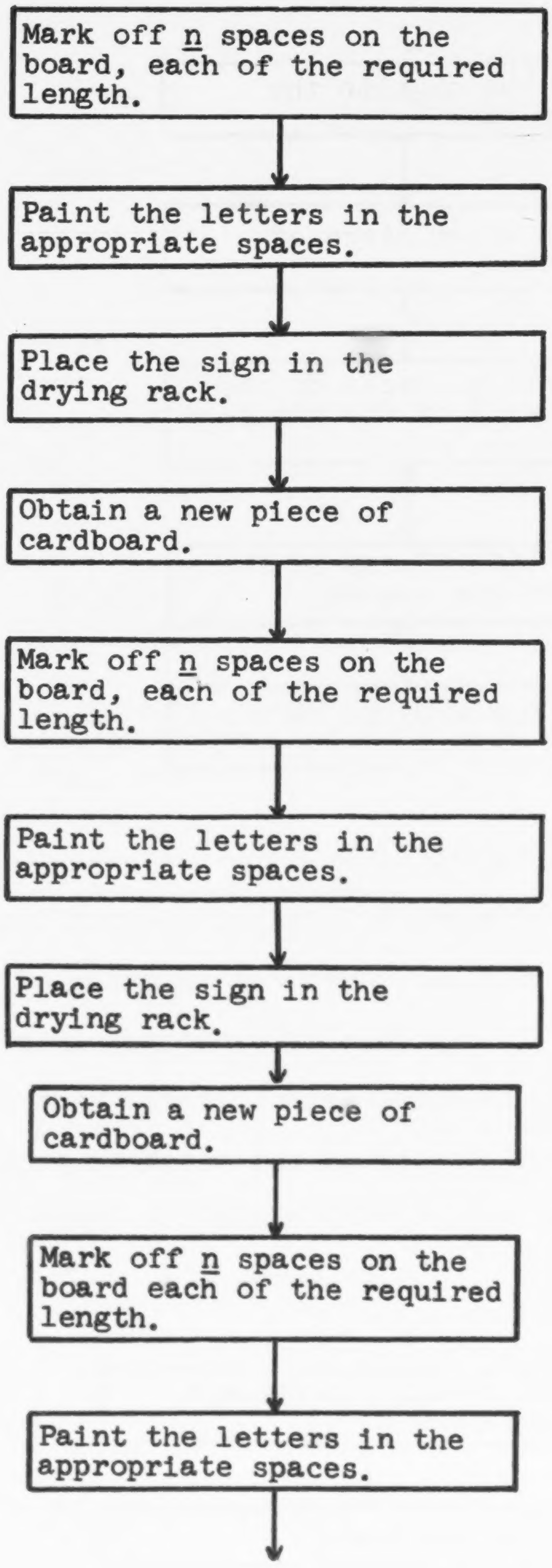
    (c) $(-8) + (36) = -28$.

8. Write a flow chart similar to that in Figure 14.8 of an algorithm for computing the difference $b - c$ of two integers $\underline{b}$, $\underline{c}$.

14.5 Iterative Algorithms

Imagine now that the novice sign painter of Section 14.1 received an order for 5 identical PLAN AHEAD signs. He could, if he chose, plan the entire job by preparing a flow chart similar to that of Figure 14.9.


Ruler, pencil, cardboard paint, brush.
Measure the board.
Count the number, n, of letters and blanks in the sign.
Divide the length of the board by the number of spaces needed to find the length of each space.
Mark off n spaces on the board each of the required length.
Paint the letters in the appropriate spaces.
Place the sign in the drying rack.
Obtain a new piece of cardboard.



Mark off n spaces on the board, each of the required length.

Paint the letters in the appropriate spaces.

Place the sign in the drying rack.

Obtain a new piece of cardboard.

Mark off n spaces on the board, each of the required length.

Paint the letters in the appropriate spaces.

Place the sign in the drying rack.

Obtain a new piece of cardboard.

Mark off n spaces on the board each of the required length.

Paint the letters in the appropriate spaces.

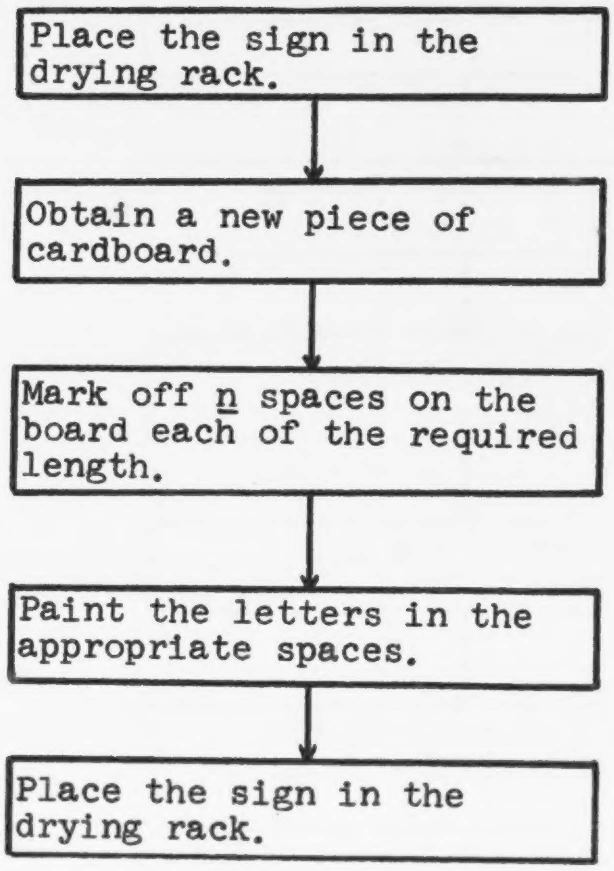
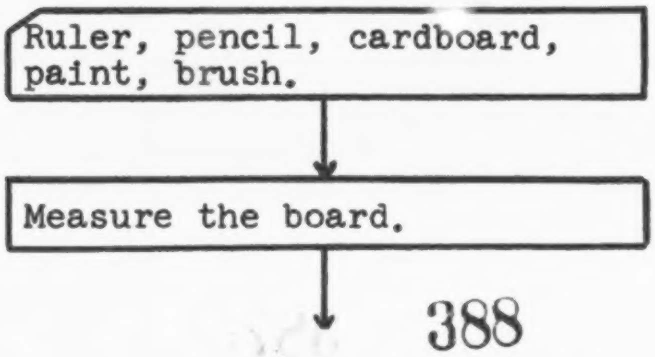

Place the sign in the drying rack.
Obtain a new piece of cardboard.
Mark off n spaces on the board each of the required length.
Paint the letters in the appropriate spaces.
Place the sign in the drying rack.

Figure 14.9

While the chart in Figure 14.9 is not too bad for an order of 5 signs, certainly a similar chart for an order of 50 signs would be far too long. In the case of processes which are to be repeated many times, it is convenient to refer back to that part of the original chart that describes the repeated process. Thus, to paint many signs, the painter could use the chart in Figure 14.10.


Ruler, pencil, cardboard, paint, brush.
Measure the board.

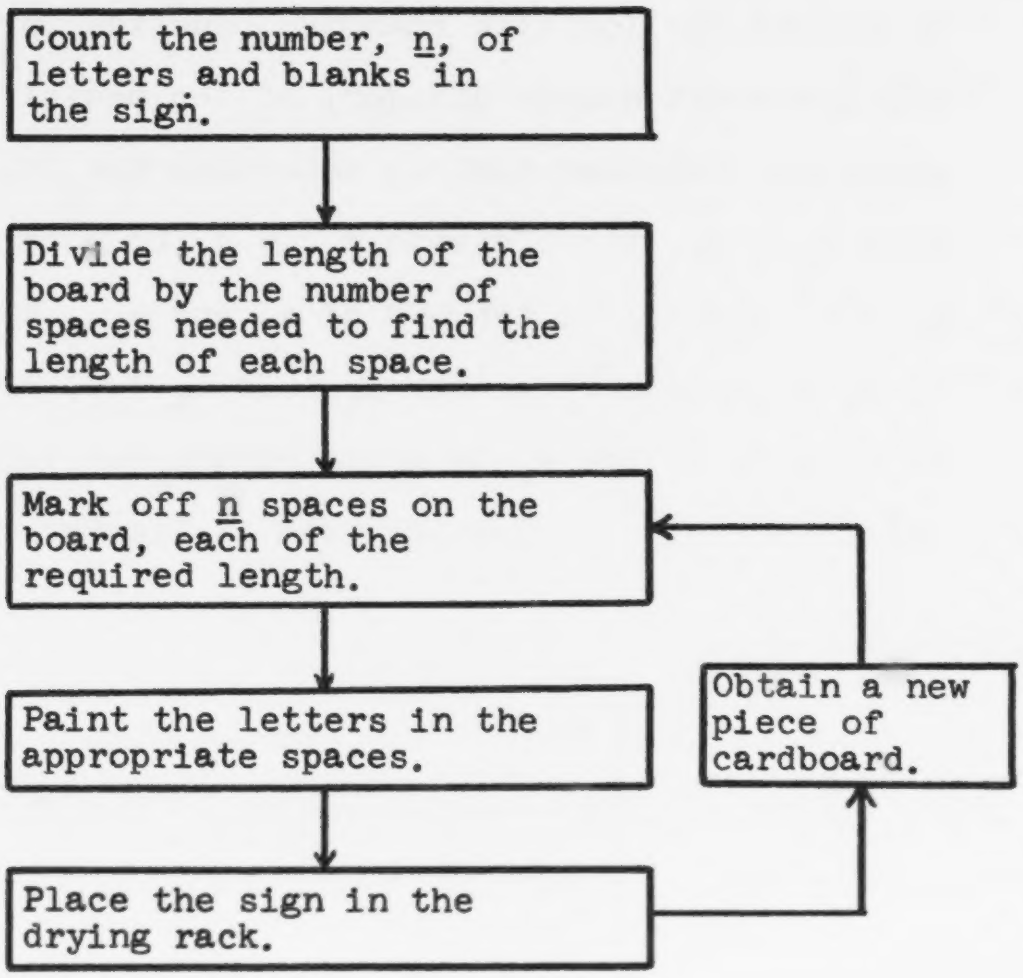

Count the number, n, of letters and blanks in the sign.
Divide the length of the board by the number of spaces needed to find the length of each space.
Mark off n spaces on the board, each of the required length.
Paint the letters in the appropriate spaces.
Place the sign in the drying rack.
Obtain a new piece of cardboard.

Figure 14.10

Flow charts of this type are said to contain a loop. The processes they represent are called iterative or repeating processes. Of course, we cannot be sure that the sign painting process ever will stop. If left alone (and if the cardboard and paint hold out), the painter could turn out signs indefinitely.

While many important mathematical routines theoretically are non-terminating, we are concerned principally with algorithms; that is, with routines which end. The sign painting process is, in a sense, an algorithm since no matter how much cardboard and paint the painter might accumulate, eventually he would run out (or perhaps die of old age). Many mathematical routines also terminate for logistic reasons. Euclid's process for finding the GCD (greatest common divisor) of two positive integers is a good example. Remember that to determine the GCD of, say, a_1 and a_2, with $a_1 \geq a_2$, first divide a_1 by a_2 to obtain remainder a_3. If $a_3 = 0$, then a_2 is the GCD of a_1 and a_2. If $a_3 \neq 0$, divide a_2 by a_3 to obtain remainder a_4. If $a_4 = 0$, then a_3 is the GCD. If $a_4 \neq 0$, divide a_3 by a_4 to obtain remainder a_5, etc. The flow chart for this routine is given in Figure 14.11.

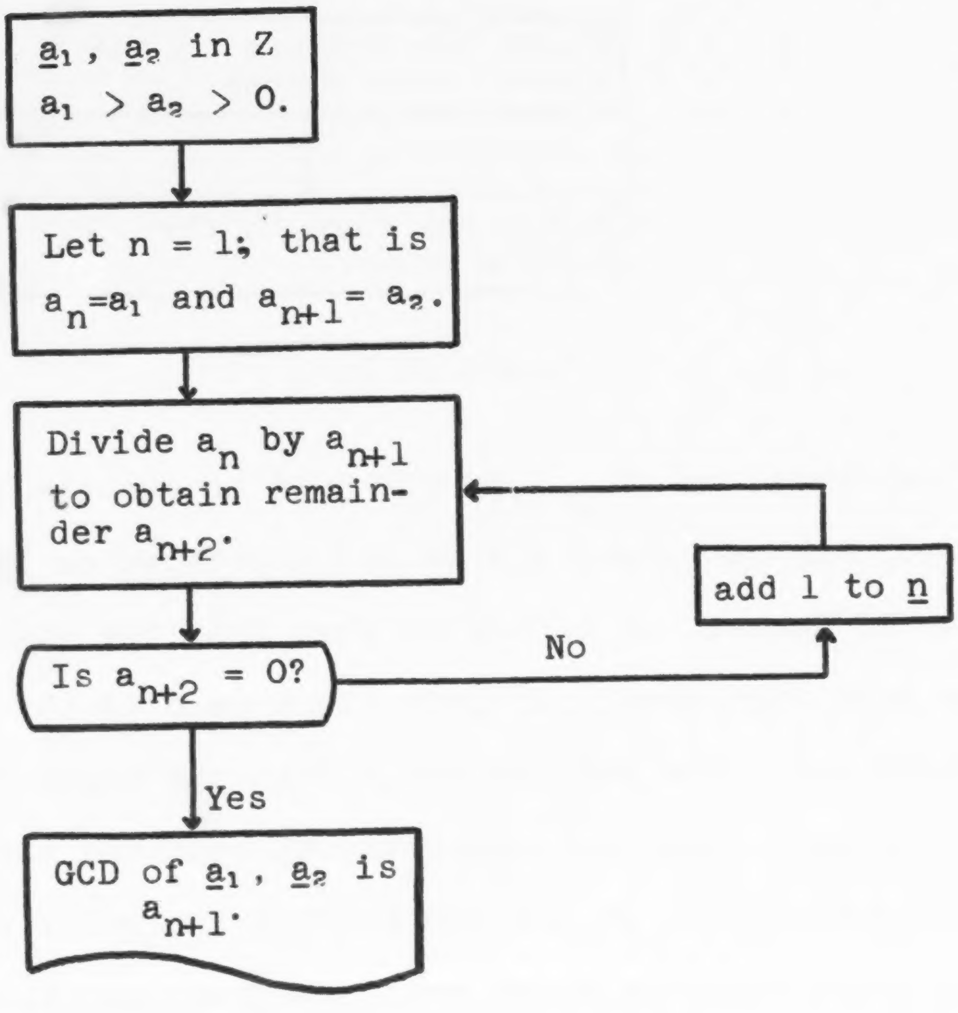

a1 , a2 in Z
a1 > a2 > 0.
Let n = 1; that is
an=a1 and an+1= a2.
Divide an by an+1 to obtain remainder an+2.
add 1 to n
Is an+2 = 0?
No
Yes
GCD of a1 , a2 is an+1.

Figure 14.11

Notice that like the chart for multiple sign production in Figure 14.10, the flow chart for Euclid's routine contains a loop. You may well ask whether Euclid's routine will terminate eventually or will continue indefinitely. To see if Euclid's process is an algorithm, that is, to see if the routine terminates, consider the following example.

Example: Find the GCD of 64 and 42 using Euclid's routine.

First iteration

$$\begin{array}{rl} 1 & \\ 42\overline{)64} & \\ 42 & \\ \hline 22 & \text{remainder} \end{array}$$

Second iteration

$$\begin{array}{rl} 1 & \\ 22\overline{)42} & \\ 22 & \\ \hline 20 & \text{remainder} \end{array}$$

Third iteration

$$\begin{array}{rl} 1 & \\ 20\overline{)22} & \\ 20 & \\ \hline 2 & \text{remainder} \end{array}$$

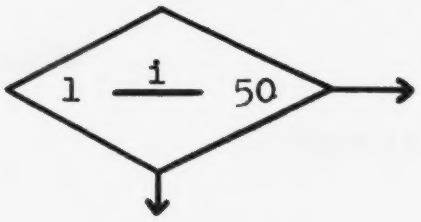
Fourth iteration

$$\begin{array}{rl} 10 & \\ 2\overline{)20} & \\ 20 & \\ \hline 0 & \text{remainder} \end{array}$$

The GCD is 2.

Since in each iteration the remainder is always less than the divisor, and since the divisor in any given iteration after the first is always the remainder in the preceding iteration, the remainders get smaller and smaller with each iteration. Since the process involves only non-negative integers, eventually a remainder of zero must be obtained. Thus, no matter how large the given numbers, Euclid's process will always produce their GCD in a finite number of iterations.

It is possible also to transform a non-terminating process into an algorithm merely by agreeing to stop after a predetermined number of interations. For example, if the painter were asked to fill an order for exactly 50 PLAN AHEAD signs, he could invent some sort of recording scheme or mechanism to tell him when he had finished the fiftieth sign. In flow charts the symbol



is commonly used for this purpose. The iteration symbol or "diamond" records the number of times a cycle has been completed and automatically channels the process out of the loop at the completion of the prescribed number of iterations. The flow chart for producing 50 PLAN AHEAD signs would look something like Figure 14.12.

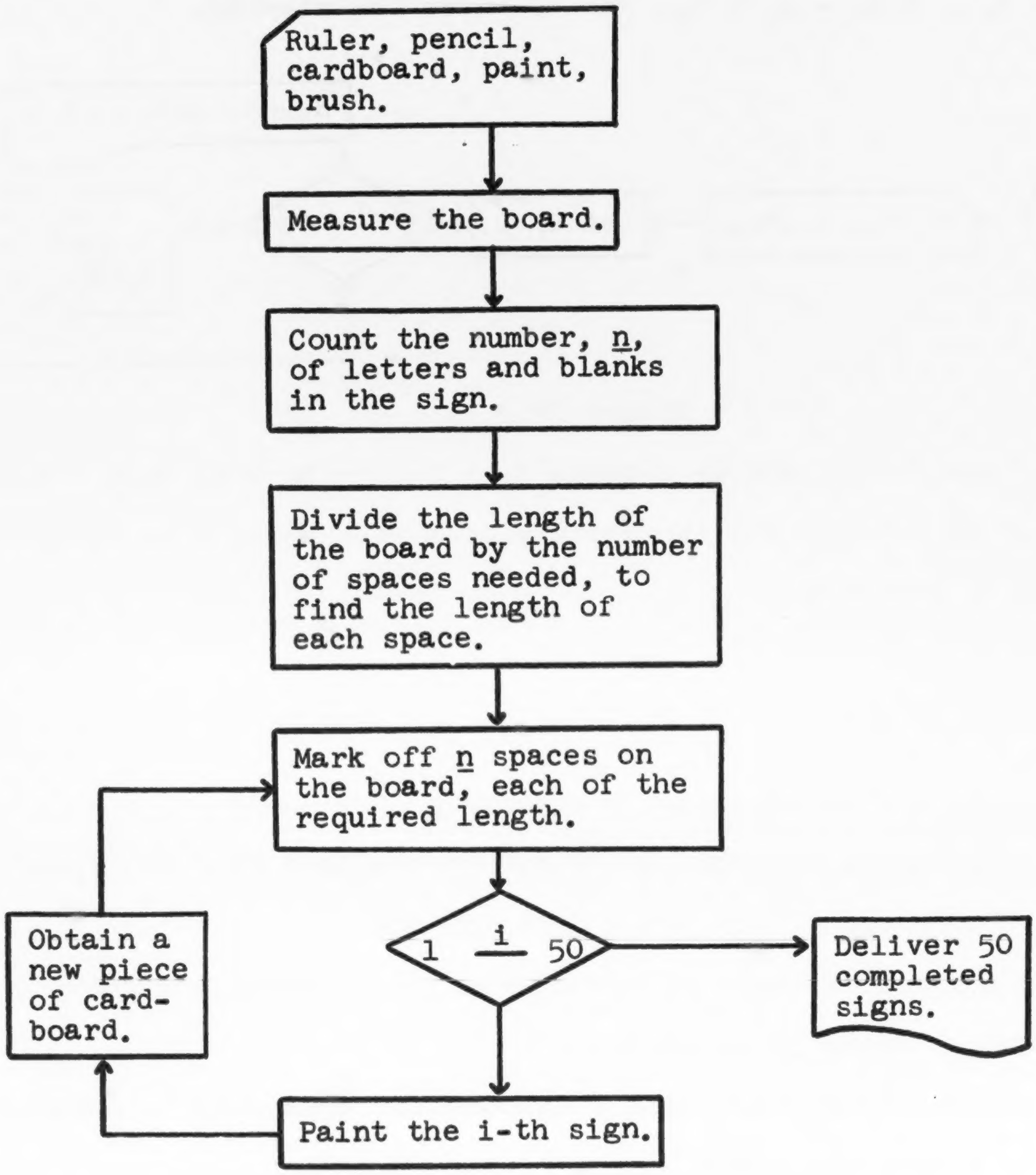

Ruler, pencil, cardboard, paint, brush.
Measure the board.
Count the number, n, of letters and blanks in the sign.
Divide the length of the board by the number of spaces needed, to find the length of each space.
Mark off n spaces on the board, each of the required length.
1
i
50
Deliver 50 completed signs.
Obtain a new piece of cardboard.
Paint the i-th sign.

Figure 14.12

As a second example, consider the procedure for finding the sum of 100 numbers $a_1, a_2, a_3, \ldots, a_{100}$. The flow chart for this algorithm is shown in Figure 14.13.

We begin by letting the zero-th subtotal be 0, and then enter the diamond. With $i = 1$, compute $S_1 = S_{1-1} + a_1 = S_0 + a_1 = 0 + a_1 = a_1$, and re-enter the diamond. With $i = 2$, compute

$S_2 = S_{2-1} + a_2 = a_1 + a_2$, and re-enter the diamond.

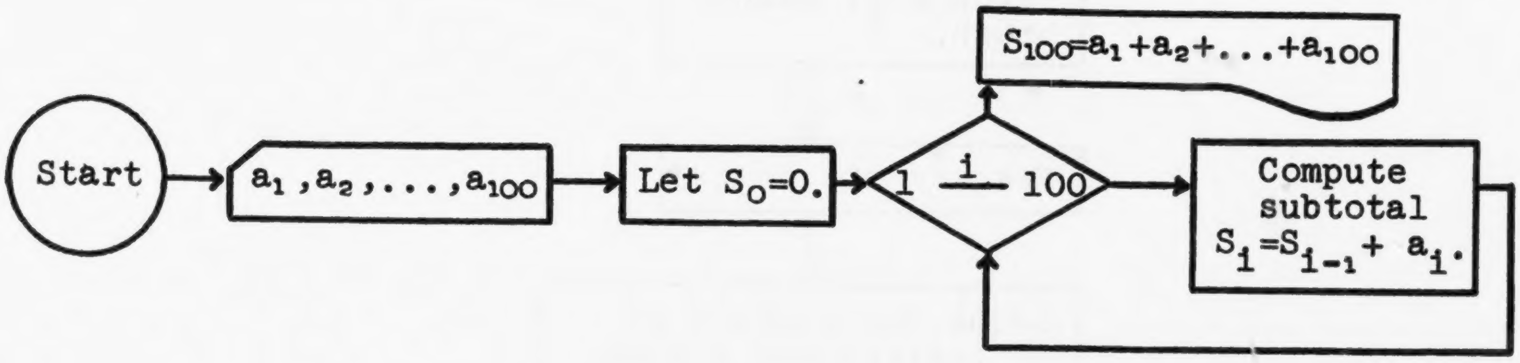

Start
$a_1, a_2, \ldots, a_{100}$
Let $S_0 = 0$.
1
i
100
$S_{100} = a_1 + a_2 + \ldots + a_{100}$
Compute subtotal $S_i = S_{i-1} + a_i$.

Figure 14.3

This time i = 3, and we compute $S_3 = S_{3-1} + a_3 = a_1 + a_2 + a_3$, and so on. After 100 iterations, S_{100} will be $a_1 + a_2 + \ldots + a_{100}$, as required.

14.6 Exercises

1. Write a flow chart for computing n' .
2. Show that a diamond can always be replaced by a combination of operation boxes and a decision box.
3. Write a flow chart with a diamond for finding the smallest number among 50 integers.
4. Write a flow chart with a diamond for arranging 50 numbers in increasing order.

14.7 Truncated Routines and Truncation Criteria

Some processes, while essentially non-terminating, can be cut off or truncated to produce algorithms. The cut-off point is determined by means of some truncation criterion. For example, the sign painter's drying rack may hold a maximum of, say, 55 signs. Thus, to fill an order of 50 signs, the painter could stop painting when the rack was full. This would give him a safety margin of 5 extra signs just in case some signs were smeared or bent before delivery. In this case his cut-off criterion would be "Is the rack full?" A flow chart for a truncated sign painting routine is shown in Figure 14.14.

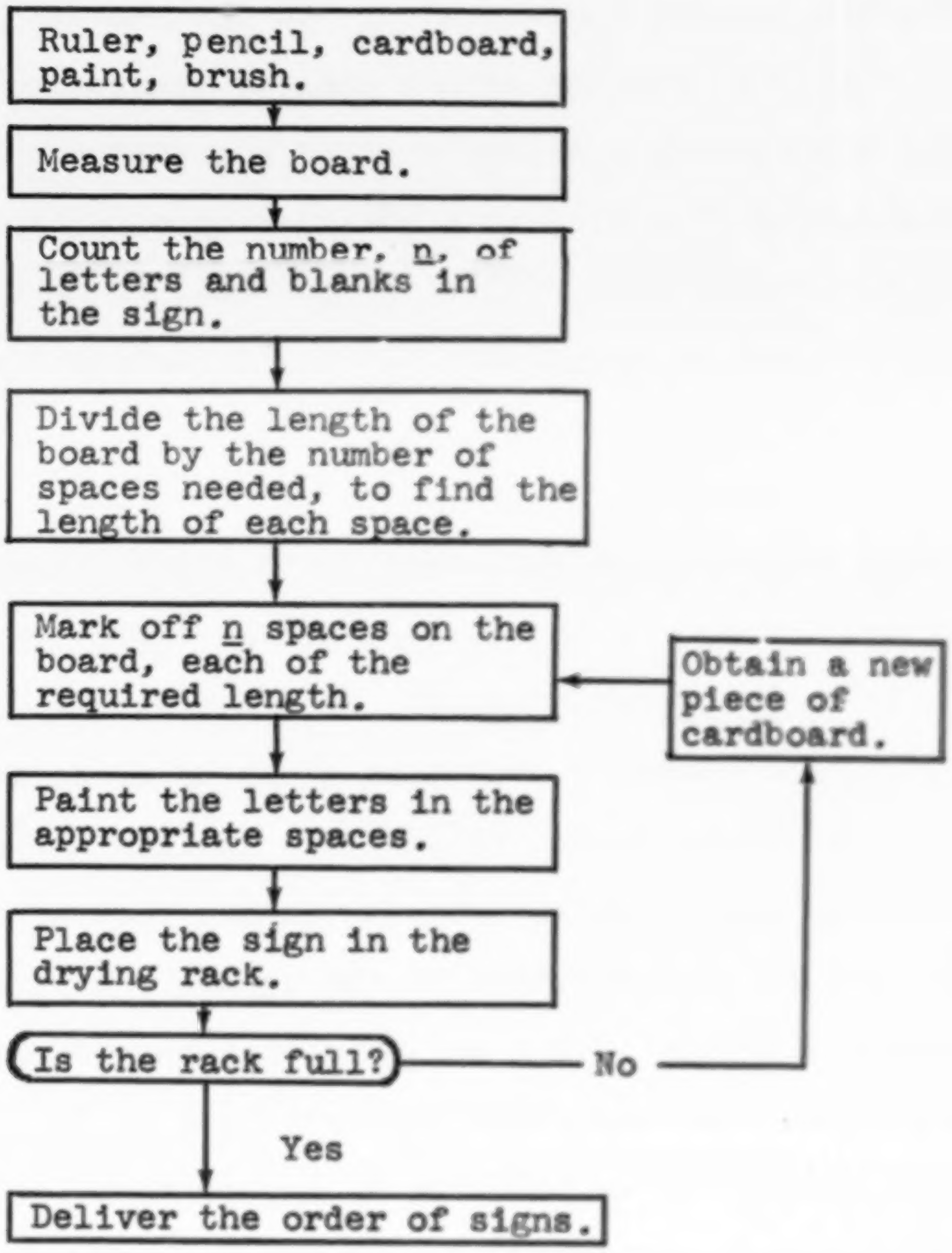

Ruler, pencil, cardboard, paint, brush.
Measure the board.
Count the number, n, of letters and blanks in the sign.
Divide the length of the board by the number of spaces needed, to find the length of each space.
Mark off n spaces on the board, each of the required length.
Obtain a new piece of cardboard.
Paint the letters in the appropriate spaces.
Place the sign in the drying rack.
Is the rack full?
No
Yes
Deliver the order of signs.

Figure 14.14

The truncation criterion is recorded in a decision box, since we use it to decide whether or not to stop the rountine.

There are many examples of truncated routines in elementary mathematics. The procedure for dividing one number by another, and carrying out the quotient as a decimal, is a non-terminating routine which we truncate according to some criterion such as "to the nearest hundredth" or "to the nearest ten-thousandth."

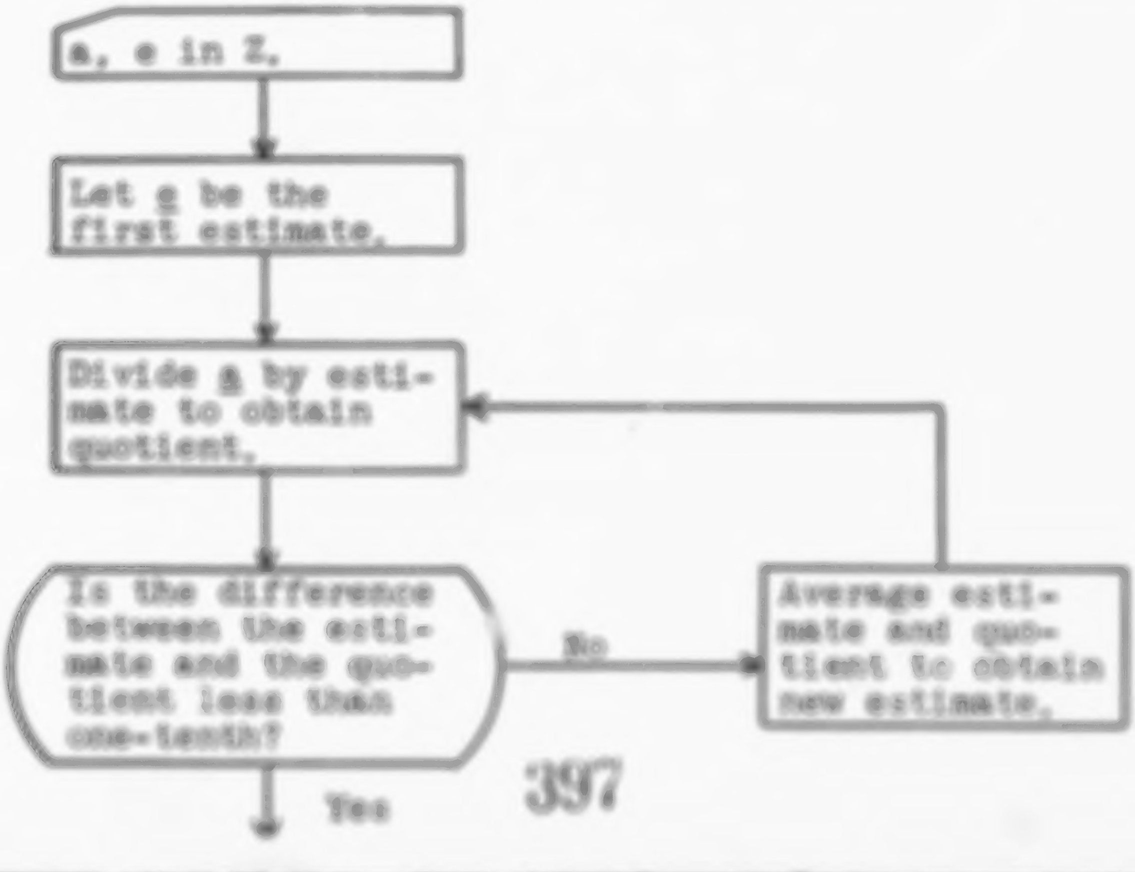
Another routine with a similar truncation criterion is useful for finding a square root of a positive number. You know that since $3^2 = 9$, 3 is a square root of 9; and since $6^2 = 36$, 6 is a square root of 36. In general, if $a, b > 0$, then b is a square root of a, written $b = \sqrt{a}$, if and only if, $b^2 = a$.

Some square roots, such as $\sqrt{9}$, $\sqrt{36}$ and $\sqrt{100}$ are easy to find. Others are more difficult; for example $\sqrt{13.7641} = 3.71$. Still other square roots are, in a sense, impossible. For example, there is no rational number whose square is 2; that is, $\sqrt{2}$ is not a rational number. It is possible, however, to find a rational number q such that q^2 is as close as we wish to 2.

Sir Isaac Newton (1642-1727) devised a routine for obtaining rational approximations to the square root of any positive number. To find an approximation to the square root of, say, 2673, using Newton's method, first estimate what an approximation to $\sqrt{2673}$ might be. Let our first approximation be 60. (Sixty is not a good guess, but we will use it anyhow to illustrate that Newton's routine does not depend on the accuracy of any approximation.) If 60 were a good approximation to $\sqrt{2673}$, divide 2673 by 60 and obtain the quotient 44. Since 60 is much greater than 44, we conclude that 60 is an overestimate, and that a better approximation must lie between 44 and 60. Now average 44 and 60, and use this average, 52, as the second approximation to $\sqrt{2673}$. If 52 is a good approximation to $\sqrt{2673}$, then $52^2 \approx 2673$, or $52 \approx \frac{2673}{52}$. Dividing 2673 by 52, we obtain the quotient 51.4. To get a third, and still better approximation to $\sqrt{2673}$, average the second approximation, 52, and the quotient, 51.4, to obtain 51.7. Now divide 2673 by 51.7 and obtain 51.7. If an approximation to $\sqrt{2673}$ were desired "correct to the nearest tenth" then this is the cut-off point. (See Section 14.8, Exercise 6.) The truncation criterion could have been "Is the difference between the estimate and the quotient less than one-tenth?" The third estimate is 51.7 and the third quotient is 51.7; hence the criterion is satisfied, and the routine stops.

A flow chart for Newton's routine is shown in Figure 14.15.

While the flow chart in Figure 14.15 will suffice for your own use, it is probably too abbreviated to be useful with a computer.


a, e in Z.
Let e be the first estimate.
Divide a by estimate to obtain quotient.
Is the difference between the estimate and the quotient less than one-tenth?
No
Average estimate and quotient to obtain new estimate.
Yes

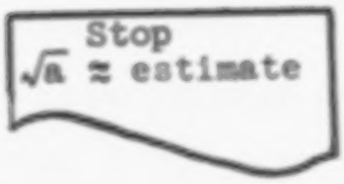
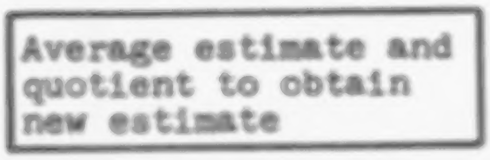

Stop
$\sqrt{a} \approx$ estimate

Figure 14.15

The operation box


Average estimate and
quotient to obtain
new estimate

lacks the detail necessary for the computer, unless the computer has already been taught to average; that is, unless a routine for averaging has already been stored in the computer. Fortunately, the routine of Section 14.2, Exercise 6 is just what is needed. To illustrate that this prior rountine is to be used here as a subroutine, we show in Figure 14.16 an altered portion of the routine in Figure 14.15.

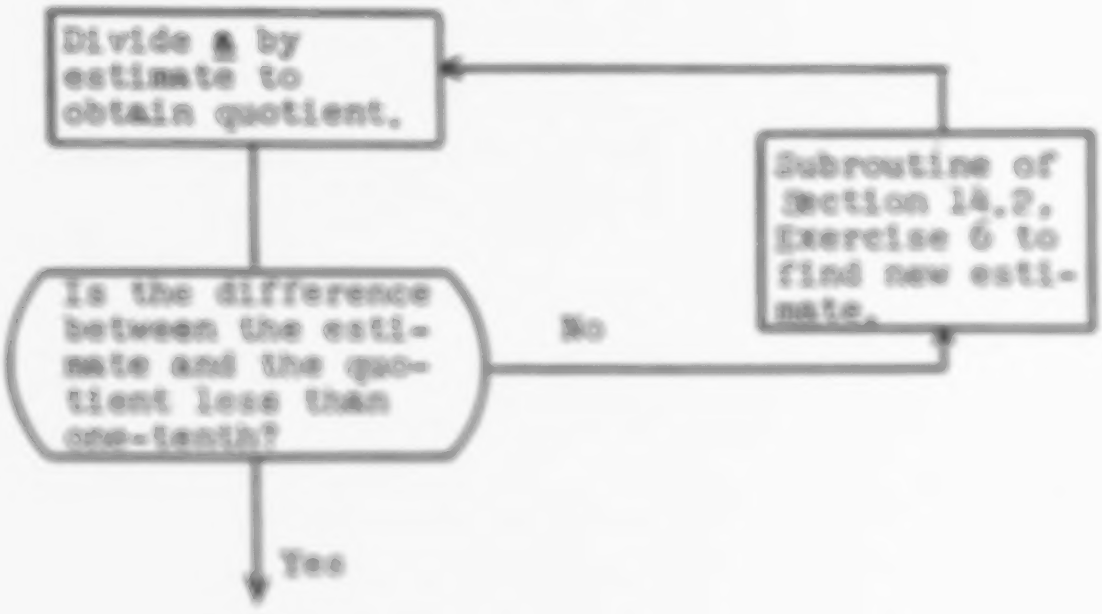

Divide a by estimate to obtain quotient.
Is the difference between the estimate and the quotient less than one-tenth?
No
Subroutine of Section 14.2, Exercise 6 to find new estimate.
Yes

Figure 14.16

Similarly, we replace the operation box

Divide a by estimate to obtain quotient.

and the decision box

Is the difference between the estimate and quotient less than one-tenth?

by appropriate subroutines, and obtain the flow chart in Figure 14.17.

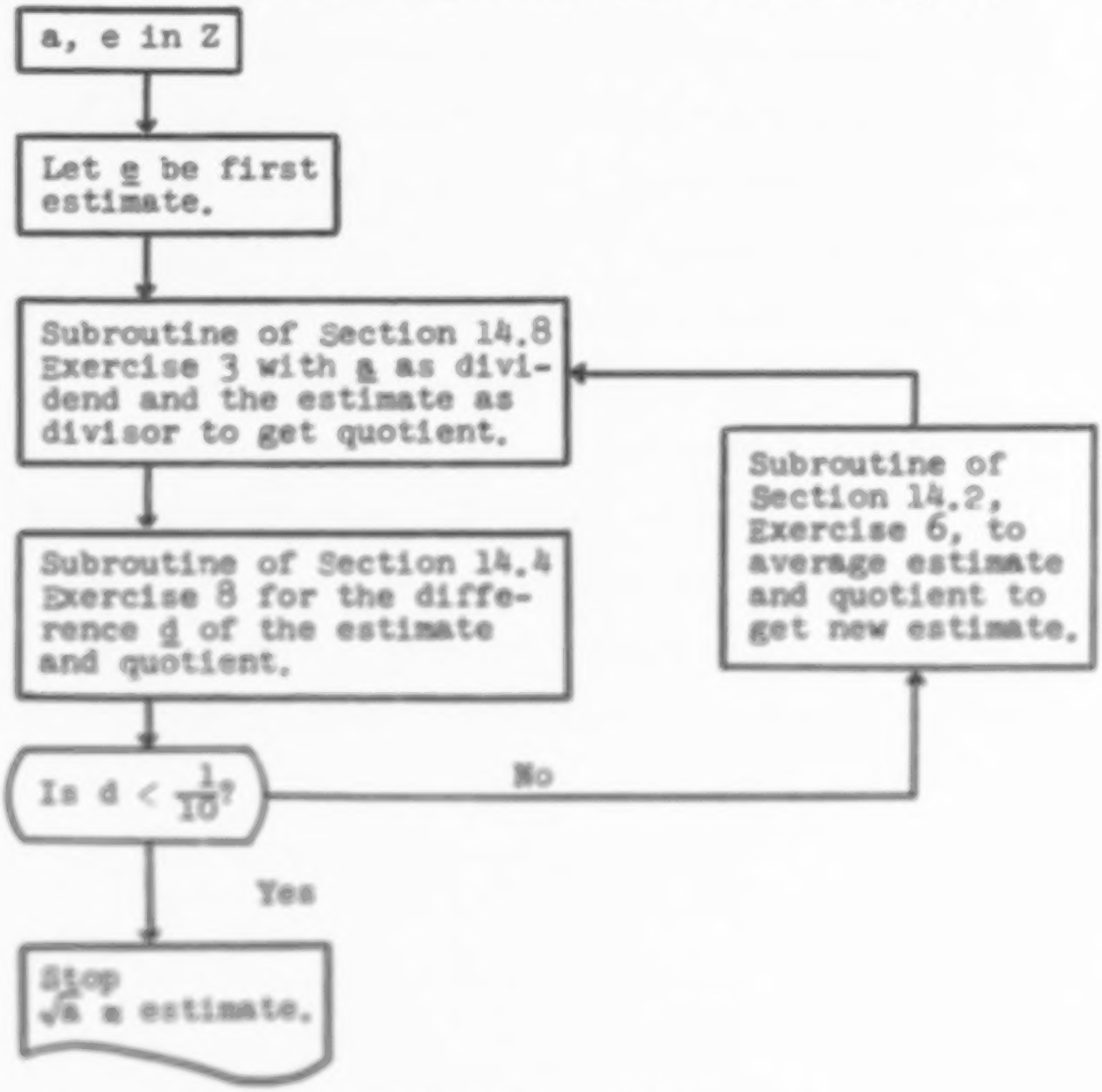
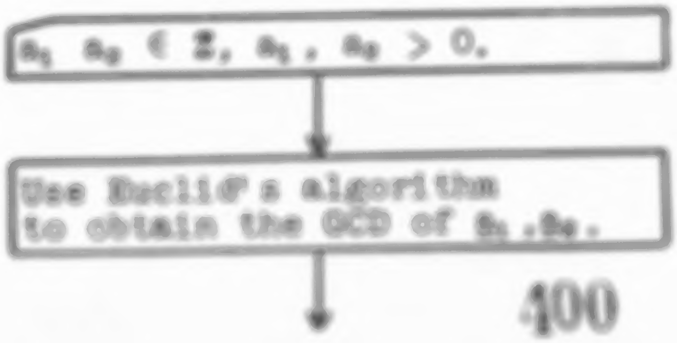

a, e in Z
Let e be first estimate.
Subroutine of Section 14.8 Exercise 3 with a as dividend and the estimate as divisor to get quotient.
Subroutine of Section 14.4 Exercise 8 for the difference d of the estimate and quotient.
Is d < 1/10?
No
Yes
Subroutine of Section 14.2, Exercise 6, to average estimate and quotient to get new estimate.
Stop
√a = estimate.

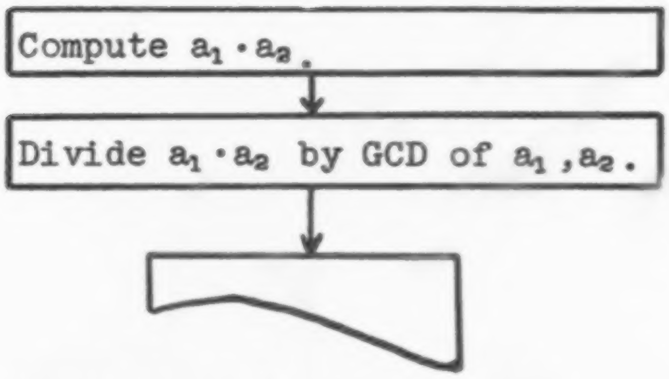
Figure 14.17

The flow chart in Figure 14.17 is satisfactory, not only for our own paper-and-pencil use, but also for the use by another computer, either human or electronic, capable of performing the routines of this chapter.

14.8 Exercises

1. Write a flow chart for averaging 50 integers which has the chart of Section 14.3 (for adding two integers) as a subroutine.
2. Write a flow chart for finding the GCD of three positive integers with Euclid's algorithm as a subroutine.
3. Write a flow chart for dividing one integer by another "to the nearest tenth." Explain why the truncation criterion is necessary here.
4. Imagine that a grasshopper is 1 unit from a grain of wheat. On his first jump he lands $\frac{1}{2}$ unit from the grain, on his second jump he lands $\frac{1}{4}$ unit from the wheat. In general, on his $\underline{n}$-th jump he lands $(\frac{1}{2})^n$ units from the wheat. Write a flow chart for summing the terms of the sequence, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, ..., and state a suitable truncation criterion.
5. Given positive integers $\underline{a}_1$ and $\underline{a}_2$ what does the following routine do?


$a_1, a_2 \in Z, a_1, a_2 > 0.$
Use Euclid's algorithm to obtain the GCD of a_1, a_2.


Compute $a_1 \cdot a_2$.
Divide $a_1 \cdot a_2$ by GCD of a_1, a_2.

6. Let e be an approximation to $\sqrt{a}$ obtained by Newton's method. Use the inequality $e < |\sqrt{a} + e|$ to show that if $|\frac{a}{e} - e| < \frac{1}{10}$, then $|\sqrt{a} - e| < \frac{1}{10}$.

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